ISSN 2427-4577

BULLETIN N° 257
ACADÉMIE EUROPÉENNE INTERDISCIPLINAIRE DES SCIENCES

INTERDISCIPLINARY EUROPEAN ACADEMY OF SCIENCES

Lundi 2 Mai 2022 (en format mixte présence-distance) :

15h30 : Conférence

« Vers une médecine computationnelle de précision »

Par **Mickaël GUEDJ**
Directeur de la biométrique,
des sciences des données et de la décision
chez Nanobiotix

Notre Prochaine séance aura lieu le lundi 13 Juin 2022 de 15h00 à 18h00
Salle Annexe Amphi Burg
Institut Curie, 13 rue Lhomond - 75005 Paris

Elle sera consacrée, à 15h00, à un hommage à notre regretté collègue Alain STAHL, décédé le 27 Avril dernier, puis à 15h30, à la conférence invitée :

« La théorie mathématique de la viabilité au service de la gestion durable : une vision différente de l'économie »

Par **Isabelle ALVAREZ**
Ingénieur en Chef des Ponts, des Eaux et des Forêts
Chercheuse au Laboratoire LISC de l'INRAE
'

Académie Européenne Interdisciplinaire des Sciences
Siège Social : 5 rue Descartes 75005 Paris
Nouveau Site Web : http://www.science-inter.com

ACADÉMIE EUROPÉENNE INTERDISCIPLINAIRE DES SCIENCES
INTERDISCIPLINARY EUROPEAN ACADEMY OF SCIENCES

Mai 2022

N°257

TABLE DES MATIERES

Prochaine séance : lundi 13 Juin 2022 de 15h00 à 18h00

- **15h00 : Hommage à notre regretté collègue Alain STAHL, décédé le 27 Avril,**
- **15h30 : Conférence**

« La théorie mathématique de la viabilité au service de la gestion durable : une vision différente de l'économie »

Par **Isabelle ALVAREZ**
Ingénieur en Chef des Ponts, des Eaux et des Forêts
Chercheuse au Laboratoire LISC de l'INRAE

ACADÉMIE EUROPÉENNE INTERDISCIPLINAIRE DES SCIENCES
INTERDISCIPLINARY EUROPEAN ACADEMY OF SCIENCES

Séance du Lundi 2 Mai 2022 mixte présentiel-distanciel

La séance est ouverte à 15h30, **sous la Présidence d'Edith PERRIER**

- **Étaient présents physiquement nos Collègues membres titulaires** de Paris : Gilbert BELAUBRE, Jean BERBINAU, Éric CHENIN, Françoise DUTHEIL, Irène HERPE-LITWIN, Jacques PRINTZ, Denise PUMAIN, René PUMAIN, Jean SCHMETS et Jean-Pierre TREUIL.
- **Etaient présent physiquement notre Collègue membre correspondant :** Benoît PRIEUR.
- **Étaient connectés à distance nos Collègues** : François BOUCHET, Ernesto DI MAURO, Jacques FLEURET et Abdel KENOUFI.
- **Était excusé notre Président** : Victor MASTRANGELO.

I. Conférence du Dr Mickaël GUEDJ : *Vers une médecine computationnelle de précision*

1. Présentation du Conférencier par notre Vice-Présidente Edith PERRIER

Mickael Guedj est diplômé de l'INSA-Lyon en Bioinformatique & Modélisation et a une thèse en Statistique Génétique réalisée au sein de la Génopole d'Evry. Il a passé ensuite 2 ans à la Ligue Nationale contre le Cancer, 8 ans chez Pharnext, 4 ans chez Servier et depuis 6 mois chez Nanobiotix, spécialisé dans le traitement des biomédicales à grande échelle pour soutenir la découverte et le développement de médicaments. Il a un intérêt particulier pour la compréhension des mécanismes physiopathologiques, l'identification de nouvelles cibles thérapeutiques et le repositionnement de médicaments.

Depuis 2021, il est Directeur de la biométrique, des sciences des données et de la décision, chez Nanobiotix.

Publications :

1) **Artificial intelligence-enhanced drug design and development: Toward a computational precision medicine** paru chez Drug Discovery Today par Philippe MOINGEON et al. accessible sur le site https://www.sciencedirect.com/science/article/abs/pii/S1359644621003962
2) **A new molecular classification to drive precision treatment strategies in primary Sjögren's syndrome**, paru chez Nature Communications par Perrine SORET et al. accessible sur le site https://doi.org/10.1038/s41467-021-23472-7
3) **Network-based repurposing identifies anti-alarmins as drug candidates to control severe lung inflammation in COVID-19** par Emiko DESVAUX et al. paru chez PLOS ONE accessible sur le site https://doi.org/10.1371/journal.pone.0254374

2. Résumé de la conférence

« *Vers une Médecine Computationnelle de Précision* »

Résumé

Vers une Médecine Computationnelle de Précision
Les Sciences Computationnelles (incluant l'Intelligence Artificielle) s'appuient sur une convergence de technologies offrant des synergies avec les technologies des Sciences de la Vie afin de capturer la valeur de données biomédicales massives sous la forme de modèles prédictifs et soutenant la prise de décision. Les algorithmes optimisent ainsi la découverte et le développement de médicaments en améliorant notre compréhension de l'hétérogénéité des maladies, en identifiant les voies moléculaires dérégulées, les cibles thérapeutiques et les médicaments candidats. Ce niveau de connaissances sans précédent concernant les spécificités des patients favorise l'émergence d'une Médecine Computationnelle de Précision permettant la conception de thérapies adaptées aux singularités de chaque patient en termes de physiologie et de caractéristiques de la maladie.

Abstract

Toward a Computational Precision Medicine
Computational Sciences including Artificial Intelligence (AI) relies upon a convergence of technologies with further synergies with Life sciences technologies to capture the value of big biomedical data in the form of predictive models supporting decision-making. Algorithms enhance drug discovery and development by improving our understanding of disease heterogeneity, identifying dysregulated molecular pathways, therapeutic targets and drug candidates. This unprecedented level of knowledge on patient specificities is fostering the emergence of a Computational Precision Medicine allowing the design of therapies tailored to the singularities of individual patients in terms of their physiology and disease features.
Mickael Guedj has 20 years experience in the treatment of large-scale biomedical data to support drug discovery & development. With a special interest for understanding pathophysiological mechanisms, new therapeutic target identification and drug repurposing.

3. Exposé du conférencier et échanges avec l'auditoire :

L'enregistrement intégral de la conférence et des échanges qui ont suivi, ainsi que celui de la présentation du conférencier par notre Vice-Présidente, Edith PERRIER, sont disponibles sur le site de l'AEIS, à la page des comptes rendus des séances mensuelles.

Nous vous proposons ci-dessous un bref compte-rendu, établi par notre collègue Jacques PRINTZ.

Compte rendu de la conférence de Mickael GUEDJ et des échanges qui ont suivi, 2 mai 2022 :

L'exposé de Mickael GUEDJ (MG), *Vers une Médecine Computationnelle de Précision*, a porté plus sur les applications de la bioinformatique que sur cette nouvelle discipline comme telle, laquelle joue bien évidemment un rôle majeur dans cette « médecine de précision ».

MG fait remonter la discipline au début du 20^{e} ou 21^{e} siècle avec le recours aux méthodes statistiques et les premières utilisations de tests statistiques χ^2 ; toutefois le vrai décollage se fait à l'occasion du projet *Human*

Genome, dans les années 2000, dont il nous informe que le décodage complet, à 100%, vient d'être obtenu cette année, soit 20 ans de travail.

Il nous précise que la « santé » peut/doit se définir par un état d'équilibre entre les différents processus à l'œuvre dans l'organisme. Compte tenu de la complexité de cet état d'équilibre et des processus en interactions, il y a beaucoup de données à manipuler, dont celle du génome. Il faut souligner que seulement 2% du génome concernent les gènes et les protéines, les 98% restant étant qualifiés de « non codants », et leur fonction mal connue ; il est donc difficile de le décrire. Le principal problème est l'interprétation de ces données ; celle-ci nécessite beaucoup de travail de modélisation relevant des technologies dites *Big Data*, car un seul génome contient environ 3,4 Md de bases ATCG. Si on veut travailler sur une cohorte de 1.000 génomes, cela fait une très grosse base de données dont on sait depuis les années 1990 que les manipulations vont nécessiter une algorithmique ad hoc compte tenu des problèmes de complexité combinatoire à prendre en compte.

MG consacre une bonne partie de son exposé à présenter les grandes lignes des outils et méthodes actuellement disponibles : données, algorithmique avec les *Computational Tool Box*, la modélisation, essentiellement statistique, avec les problèmes associés comme l'analyse dite « en grande dimension » qui reste un défi d'ordre mathématique en matière de « spatialisation » pour visualiser l'information ainsi collectée.

L'un des problèmes clé présentés par MG concerne l'analyse et la classification des interactions, d'où les problématiques de graphes de connaissances, de réseaux d'inférences, qui sont analysées avec les méthodes développées pour le *Deep Learning*, en signalant au passage certaines difficultés et/ou biais comme le sur-apprentissage. Toutes ces analyses sont faites grâce à des plates-formes intégrées – *All-in-one Computing platforms* – qui font partie de l'outillage standard actuellement disponibles.

La dernière partie de l'exposé de MG est consacrée aux applications comme l'analyse des cancers du sein, la rétro-analyse – *repurposed drugs* – qui consiste à repartir des gènes pour remonter aux maladies résultant des dysfonctionnements de ces gènes, par rétro-conception, et enfin l'analyse des patrimoines génétiques, pour aborder des problématiques plus globales comme celles du covid-19. Il ne traitera en détail que cette dernière application.

Dans sa conclusion, MG donnera 6 points d'amélioration imputables à ces nouveaux outils et méthodes :

1. Meilleure vitesse d'exécution
2. Réduction des coûts
3. Augmentation de la diversité moléculaire
4. Simplification des essais cliniques, demandant moins d'animaux de laboratoire
5. Moins de patients requis,
6. Meilleure efficacité globale.

Sa dernière planche, ci-dessous, mérite une mention spéciale car elle illustre particulièrement bien les problématiques que nous souhaitons aborder dans notre colloque futur sur l'interdisciplinarité car derrière chacune des disciplines mentionnées il y a des communautés dont il faut organiser les interactions, et pour cela trouver un langage commun.

Questions/réponses

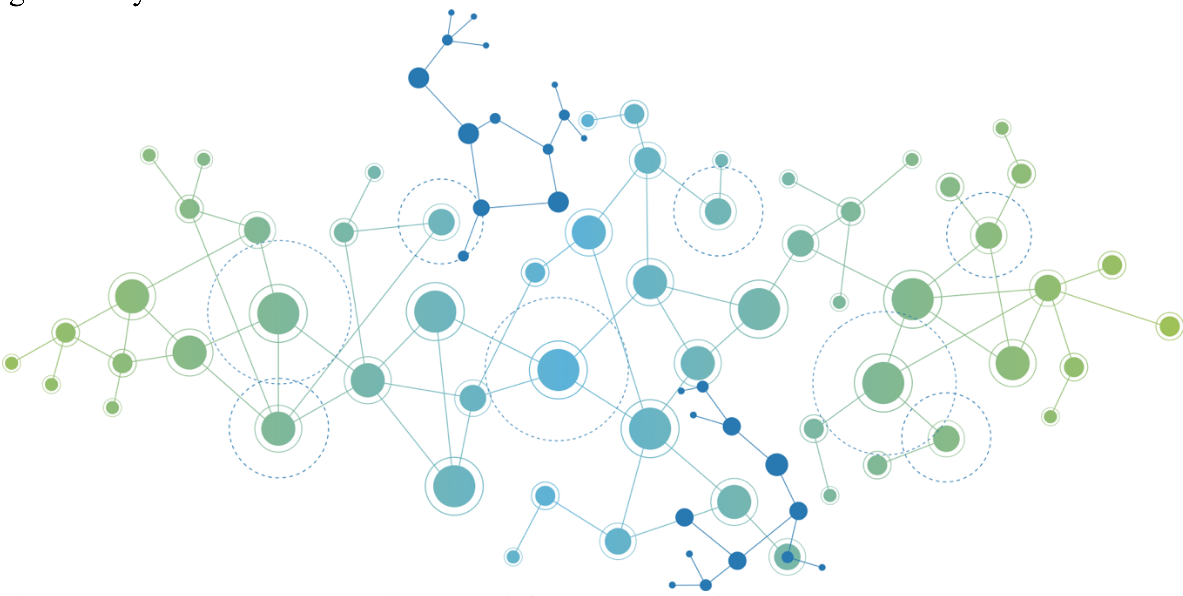
Plusieurs questions sont soulevées par l'auditoire in situ et à distance sur les thèmes :

- *Drug design*
- Interdisciplinarité et complexité
- Standardisation des données et des plates-formes pour faciliter les échanges, par analogie avec ce que fait le CERN depuis longtemps
- Modèles et *Big data* : comment calibrer tout cela en évitant les corrélations fallacieuses

NB du rédacteur ~~JPZ~~ Jacques PRINTZ: *sur ce dernier point, plusieurs interrogations sont soulevées concernant l'usage statistique des données en référence à un article de C. Anderson, dans la revue* WIRED, *The End of Theory: The Data Deluge Makes the Scientific Method Obsolete dont je donne ici la référence précise :*
[https://www.wired.com/2008/06/pb-theory/?msclkid=f315ccfdcf6f11eca3f40678feaa0d8f].
Cet article a eu et continue à avoir une audience importante dans le grand public peu informé de la méthode scientifique, bien que l'argument soit totalement faux ! il est donc important d'en avoir connaissance pour mieux le réfuter et expliquer la réalité de la modélisation.

Dernière planche de MG :

Elle illustre particulièrement bien la nécessité de l'approche interdisciplinaire où les communautés en interactions doivent définir le langage commun – sujet considéré comme difficile – qui leur permet de se comprendre et d'agir de façon cohérente au sein des projets, un thème abordé dans une conférence précédente avec l'ingénierie système.



En complément, MG nous recommande son article, en coopération :
Philippe Moingeon, Melaine Kuenemann, Mickaël Guedj *: Artificial intelligence-enhanced drug design and development: Toward a computational precision medicine.* Drug Discovery Today. 2022

Sont également disponibles dans la rubrique Comptes-rendus des conférences mensuelles du site de l'AEIS http://www.science-inter.com les enregistrements en visio-conférences et les documents suivants pour la conférence de Mickaël GUEDJ :

- Conférence (.m4v)
- Présentation du conférencier (.m4v)
- Résumé (.pdf)
- Synthèse (.pdf)
- Support de présentation (.pdf)
- Support de présentation (.pptx)
- Article associé 1 (.pdf) : « *Network-based repurposing identifies anti- alarmins as drug candidates to control severe lung inflammation in COVID-19* », de Emiko Desvaux et al., *PLOS ONE, 2021.*
- Article associé 2 (.pdf) : « *Artificial intelligence-enhanced drug design and development: Toward a computational precision medicine* », de Philippe Moingeon et al., *Drug Discovery Today, 2021 in press.*
- Article associé 3 (.pdf) : « *A new molecular classification to drive precision treatment strategies in primary Sjögren's syndrome* », de Perrine Soret et al., *Nature Communications, 2021.*

REMERCIEMENTS
Nous tenons à remercier vivement M. Yann TRAN et Mme Annabelle POIRIER de l'Institut Curie pour la qualité de leur accueil.

II. Conférence d'Isabelle ALVAREZ : *La théorie mathématique de la viabilité au service de la gestion durable : une vision différente de l'économie*

Isabelle Alvarez

Chercheur à INRAE, accueillie à l'ISC-PIF. De formation ingénieur généraliste (X83, Engref), dans une première vie elle est chercheuse en Intelligence artificielle sur les problèmes d'explication de résultats, au Cemagref et au LIP6 (Thèse en 1992). Elle applique ses travaux dans le domaine agricole et environnemental. Elle s'intéresse à l'ingénierie des systèmes complexes (codirectrice du RNSC de 2014 à 2017). Et bascule de l'IA dans la théorie de la viabilité en 2010.

Résumé

La théorie mathématique de la viabilité (Aubin, 1991) étudie la compatibilité entre un système dynamique et un ensemble de contraintes. Ce formalisme permet de proposer des définitions pour les concepts liés à la durabilité et d'étudier les liens entre ces concepts, comme la robustesse et la résilience (Martin, 2004). Dans ce cadre il est possible d'éviter les arbitrages entre les préoccupations de court terme ou de long terme. Il permet aussi de prendre en compte simultanément plusieurs aspects de la durabilité. L'intérêt suscité par la théorie de la viabilité a entraîné des travaux en informatique pour rendre opérationnels les outils d'analyse proposés. C'est à l'heure actuelle, avec le changement de pratique qu'elle suggère, la principale limite à sa mise en œuvre. En effet, même si elle est rattachée au contrôle optimal, la théorie de la viabilité propose un changement de perspective par rapport à l'optimisation. Avant de se poser des questions en termes d'objectif à optimiser, on s'intéresse à la définition des états souhaitables et des moyens admissibles pour y maintenir le système étudié, ainsi qu'aux conséquences pour les évolutions possibles du système. Plusieurs exemples seront proposés pour expliquer l'intérêt de la démarche au-delà de la théorie mathématique.

Abstract

The mathematical viability theory (Aubin, 1991) offers concepts and methods that are suitable to study the compatibility between a dynamical system and constraints in the state space. It is particularly suitable to study the sustainability of socio-ecosystems. In this framework it is possible to study the links between concepts related to sustainability, such as robustness and resilience (Martin, 2004). It avoids the trade-off between short term and long-term considerations, and also between different features of sustainability. Related work in computer science has led to approximation algorithms with proved convergence, but the computation of viability kernel and regulation maps are still challenging. It is the main limitation to its implementation along with the change of vision it suggests. Actually, even if it is linked to optimal control, the mathematical viability theory proposes a change of perspective with respect to optimization. Before addressing questions in terms of objective function, it focuses on the definition of the set of desirable states and the admissible means to maintain the system in it, and studies the consequences for the possible evolution of the system. Several examples are discussed to show the interest of the approach beyond the mathematical theory.

Articles associés et références librement téléchargeables

Martin, 2004 : " The cost of restoration as a way of defining resilience: a viability approach applied to a model of lake eutrophication ". Ecology and Society 9(2): 8. http://www.ecologyandsociety.org/vol9/iss2/art8

I. Alvarez, R. de Aldama, S. Martin, R. Reuillon, 2013 : " Assessing the Resilience of Socio-Ecosystems: Coupling Viability Theory and Active Learning with kd-Trees. Application to Bilingual Societies ", 23rd International Joint Conference on Artificial Intelligence, IJCAI'13, Beijing, China, pp. 2776-2782 (2013) http://ijcai.org/Proceedings/13/Papers/409.pdf

I. Alvarez, L. Zaleski, J-B. Briot, M. Irving. 2022. " Collective management of environmental commons with multiple usages: a guaranteed viability approach ". ArXiv. https://arxiv.org/abs/2107.02684

Documents

Pour préparer la conférence d'Isabelle ALVAREZ, nous vous proposons :

p. 9, I. Alvarez, R. de Aldama, S. Martin, R. Reuillon, 2013 : " Assessing the Resilience of Socio-Ecosystems: Coupling Viability Theory and Active Learning with kd-Trees. Application to Bilingual Societies ", 23rd International Joint Conference on Artificial Intelligence, IJCAI'13, Beijing, China, pp. 2776-2782 (2013) http://ijcai.org/Proceedings/13/Papers/409.pdf

p. 16, I. Alvarez, L. Zaleski, J-B. Briot, M. Irving. 2022. " Collective management of environmental commons with multiple usages: a guaranteed viability approach ". ArXiv. https://arxiv.org/abs/2107.02684

Assessing the Resilience of Socio-Ecosystems: Coupling Viability Theory and Active Learning with kd-Trees. Application to Bilingual Societies*

Isabelle Alvarez(1,3)
(1) LIP6, UPMC
F-75005 Paris, France
isabelle.alvarez@lip6.fr

Ricardo de Aldama (2)
(2) ISC-PIF
F-75005 Paris, France
ricardo.de.aldama@iscpif.fr

Sophie Martin (3)
(3) Irstea, LISC
F-63172 Aubiere Cedex, France
sophie.martin@irstea.fr

Romain Reuillon (2)
(2) ISC-PIF
F-75005 Paris, France
romain.reuillon@iscpif.fr

Abstract

This paper proposes a new algorithm to compute the resilience of a social system or an ecosystem when it is defined in the framework of the mathematical viability theory. It is applied to the problem of language coexistence: Although bilingual societies do exist, many languages have disappeared and some seem endangered presently. Mathematical models of language competition generally conclude that one language will disappear, except when the relative prestige of the languages can be modified. The viability theory provides concepts and tools that are suitable to study the resilience, but with severe computational limits since it uses extensive search on regular grids. The method we propose considers the computation of the viability output sets as an active learning problem with the objective of restraining the number of calls to the model and information storage. We adapt a kd-tree algorithm to approximate the level sets of the resilience value. We prove that this algorithm converges to the output sets defined by the viability theory (viability kernel and capture basin). The resilience value we compute can then be used to propose a policy of action in risky situations such as migration flows.

1 Introduction

Assessing the resilience of an ecological or social system is becoming a challenge in the context of sustainable development (as stated in [Perrings, 2006]). Traditionally resilience measures the ability of a system to recover after a perturbation (see [Martin, 2004] for an analysis of operational definitions of resilience). In this context the mathematical theory of viability is an interesting framework since it studies the compatibility of dynamical systems and constraints [Aubin *et al.*, 2011]. This framework is used in sustainability studies in order to find control policies that keep the system in a given constraint set, such as the concept of tolerable windows [Bruckner *et al.*, 2003] for climate change studies. When the system evolves outside the desirable constraint set, viability theory can assess whether and how the system can be driven back to desirable states. In particular it can provide the returning time [Doyen and Saint-Pierre, 1997]. In this paper we quantify the resilience as the inverse value of this returning time, as it is done in [Martin, 2004] for ecological systems.

Viability analysis is based on the computation of the viability kernel and its capture basin. The viability kernel gathers all the states from which there exists a control function that allows the evolution to stay in the constraint set. Its capture basin gathers the states from which it is possible to reach the viability kernel in finite time. Unfortunately, the complexity of the computation task is exponential with space or time when using a discrete approximation on a grid [Saint-Pierre, 1994]. Therefore this method is limited to very low dimension (at most 3) or to linear models, so its applicability is very much impaired. Moreover, the complexity of the model in real application can limit the practical use of the method because of computation problems when running the model. Dimension 4 was exceptionally reached in a real food processing application with the help of grid computing (see [Sicard *et al.*, 2012], [Reuillon *et al.*, 2010]).

In the particular case of finite time horizon, [Bonneuil, 2006] proposes a method based on simulated annealing to generate trajectories near the boundary. This method works in higher dimensions but it only produces a set of trajectories so it cannot be used to compute the viability kernel nor resilience level sets. Following the same idea of focusing on the boundary in the general case, [Deffuant *et al.*, 2007] introduces classification functions in Saint-Pierre's viability algorithm in order to reduce the number of calls to the model and to represent the viability set in a more compact way than the traditional regular grid. But the Support Vector Machine (SVM) functions used as classification function in [Deffuant *et al.*, 2007], although very attractive, don't fulfill the conditions of the convergence theorem. So the output sets produced

*The research leading to these results has received funding from the E.C.'s FP7/2009-2013 under grant agreement DREAM n. 222654-2.

with the SVM method are not reliable. To overcome these limitations, we also consider the computation of the boundary at each step as an active learning problem, but we use kd-trees as it is done in [Rouquier *et al.*,] to limit the number of calls to the model. Using kd-trees as classification functions as well as oracle functions to store the successive approximations of the viability sets, it is possible to converge towards the true viability kernel or capture basin.

The paper is organized as follows: Section 2 presents the language competition problem and its translation in the viability framework. We define the resilience and show how it is related to the capture basin defined by the theory. Section 3 is devoted to the computation of the capture basin. It explains how it can be seen as an active learning problem, and it describes the kd-tree algorithm that is used in that purpose. It also gives the proof of convergence of the algorithm towards the output sets of the viability theory. Section 4 presents the result of the computation for the language competition problem and how the computed resilience value can be used to propose action policies that guarantee the language coexistence.

2 Resilience of bilingual societies

Bilingual societies exist, but history tells us that many languages have disappeared. Presently, many languages seem endangered. Some researchers have proposed mathematical models to study language competition (see for instance [Abrams and Strogatz, 2003]). These models are consistent with historical data from past or present endangered languages (such as Welsh, Scottish, Gaelic, Quechua). Models without bilingual population generally conclude that one language eventually becomes extinct. When a bilingual population exists, the survival of the language diversity depends on the relative prestige of languages [Bernard and Martin, 2012].

But the question of resilience of these societies is still pending. Real societies are subjected to migration: both immigration and emigration (due to economic, environmental, educational or safety reasons for instance) disturb the dynamics of the different language speakers and can jeopardize the diversity of a bilingual society. This is the reason why it is interesting to study the ability of a bilingual society to recover from such a perturbation. Here we propose to study this resilience in the viability framework.

2.1 Model of bilingual society

The model from [Bernard and Martin, 2012] describes a bilingual society considering three groups in the population: the monolingual speakers of language A, the monolingual speakers of language B, and the bilingual speakers AB. The model is described by σ_A the proportion of speakers of A, σ_B the proportion of speakers of B, and the proportion of bilinguals σ_{AB} with $\sigma_{AB} = 1 - \sigma_A - \sigma_B$. The rate at which speakers of one language switch to become speakers of the second language depends on the attractiveness of this second language, but transitions between groups concern A or $B \to AB$ and conversely. Since it is unlikely for one speaker to switch language from scratch, we have $P_{A\to B} = P_{B\to A} = 0$. We then have:

$\frac{d\sigma_A}{dt} = \sigma_{AB} P_{AB\to A} - \sigma_A P_{A\to AB}$, (with analogous equations for $\frac{d\sigma_B}{dt}$ and $\frac{d\sigma_{AB}}{dt}$).

Concerning transitions, monolinguals are sensitive to the size of the monolingual group of the other language when bilinguals are sensitive to the whole group of speakers of the other language. The proportionality factor is the prestige of one language compared to the other (we note s the prestige of language A, so $1 - s$ is the prestige of language B). A parameter a models how the attractiveness of a language scales with the proportion of speakers. So we have:

$P_{AB\to A} = (1 - \sigma_B)^a s$ and $P_{A\to AB} = \sigma_B^a (1 - s)$. The value of the prestige, s, can evolve with public action: education, advertising, incentive policy, arranged employment, positive discrimination, and so on. Since these kinds of policy take time to give results, $\frac{ds}{dt}$ is considered here as a control on the dynamics, bounded in some interval $U = [-\bar{u}; \bar{u}]$ with $\bar{u} > 0$. In this framework the model of a bilingual society is defined by the following equations:

$$\begin{cases} \frac{d\sigma_A}{dt} &= (1 - \sigma_A - \sigma_B)(1 - \sigma_B)^a s - \sigma_A \sigma_B^a (1 - s) \\ \frac{d\sigma_B}{dt} &= (1 - \sigma_A - \sigma_B)(1 - \sigma_A)^a (1 - s) - \sigma_B \sigma_A^a s \\ \frac{ds}{dt} &= u \in U \end{cases} \quad (1)$$

If s is constant in $]0; 1[$, the dynamics has three equilibria: $(0, 1)$, $(1, 0)$ which are stable, and an unstable one. Consequently, one language is doomed to become extinct. Therefore it is necessary to apply control policy on s to insure the coexistence.

2.2 Set of desirable states and viability domain

In order to ensure the sustainability of the language diversity, it is necessary to define what is considered as desirable. For instance, an easy way to define language diversity would be to consider that the proportion of monolingual speakers of each language has to be above a threshold $\underline{\sigma}$. The set of desirable set in $E = [0, 1]^3$ is then $K = \{(\sigma_A, \sigma_B, s)\}$ such that:

$$\begin{cases} 0 < \underline{\sigma} \leq & \sigma_A & \leq 1 \\ 0 < \underline{\sigma} \leq & \sigma_B & \leq 1 \\ 0 \leq & s & \leq 1. \end{cases} \quad (2)$$

We suppose now that this threshold is set to 0.2. We consider that the boundary of the control variation $\bar{u}$ is set to 0.1. Parameter a set to 1.31 according to the literature. (It is calibrated in [Abrams and Strogatz, 2003] from historical data).

In this framework the viability theory applies. It can be shown that there exists a subset of K which is a viability domain D: from each state $x = (\sigma_A, \sigma_B, s)$ in D there exists a control function $u(t)$ that allows the evolution of the dynamics 1 to stay inside D. Figure 1 shows the shape of D.

2.3 Capture basin and resilience value

Outside the viability domain D, the bilingual society may not be viable any longer. In particular, the proportion of one of the languages may fall under the threshold $\underline{\sigma} = 0.2$, which is not considered as a satisfying future. [1]

[1]Since there exists a viability domain in K, the viability theory states that there must be a viability kernel including D, but actually it

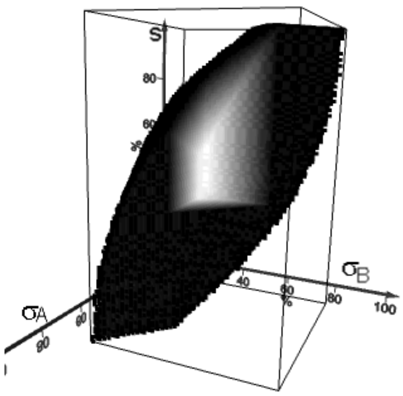


Figure 1: Viability domain D for the problem of language coexistence. The bounding box is the set of desirable states. Evolution starting in D can stay within forever.

We consider that without additional information, the viability domain D gathers the secure states, since from any starting point in D, we are sure that the evolution can remain in the set of desirable states K (that is, each language is spoken by at least 20% of the population) with the appropriate action policy.

In case of perturbation, the state of the bilingual society can jump from D to a state outside the viability domain. Here we consider that perturbations can affect each variable: Migrations change the value of the proportion of speakers of each language. But the prestige of one language may also be disturbed, for example by criminal cases or impressive achievement concerning one group. In a sustainability viewpoint, the issue is to know if it is possible to reach a secure position in the viability domain, starting from the disturbed state. The set of states from which there exists a control policy that drives the system back to D is called the capture basin of D in the viability theory.

$$Capt(D) = \{x \mid \exists u(.),\ \exists T \mid x(T) \in D\}$$

At each time horizon T, it is possible to consider the corresponding capture basin $Capt(D,T) = \{x \mid \exists u(.),\ x(T) \in D\}$. Its boundary $\partial Capt(D,T)$ is the level set of the returning time T to D. In particular we have by definition: $Capt(D,0) = D$.

Definition 1 (Resilience of a state) *The resilience of a state x outside the viability set D is $r(x) = 0$ if $x \notin Capt(D)$. Otherwise, it is $r(x) = \frac{1}{T}$ with $T = min\{t \mid x \in Capt(D,t)\}$.*

Definition 2 *The resilience of a bilingual society to a given perturbation set P is: $min\{r(x+y) \mid x \in D,\ y \in P\}$.*

When the capture basin is not greater than the viability domain, then the resilience of the bilingual society is zero: In case of a perturbation, the proportion of speakers of one language will never recover. If the capture basin is greater than the viability domain D, that means that if the state of the society is not as expected yet but still in the capture basin, it can return to the set of desirable states with an appropriate control policy. Computing the capture basin of D will allow to measure resilience and to find these control policies.

could be D itself. In this latter case, outside D, one of the languages is doomed to be spoken by less than a proportion $\underline{\sigma} = 0.2$ of the population.

3 Computation of the resilience

3.1 Computation of the capture basin: active learning sampling with kd-tree

The computation of the capture basin of the viability domain D can be seen as a particular problem of viability [Aubin, 2001]. Actually, the capture basin can be defined as a projection of the viability kernel of an auxiliary viability problem with one additional dimension, the time dimension. So the same computational limits apply. The classical approach consists in sampling the state space along a grid, and store the coordinates of all viable points, but this is costly in terms of time and memory.

We consider the dynamical model as a black box F defined from $E = [0,1]^3$ to $E = [0,1]^3$. F is the discrete dynamical system corresponding to the model 1. Based on this black box function F, an exact function $f_{(S)}(x)$ from $E = [0,1]^3$ to $\{0,1\}$ is built, which can tell as an oracle whether there exists or not a control $u \in U$ such that the trajectory starting at point x with control u belongs to a given set S at the following time step. If this set S is $Capt(D,T)$ then $x \in Capt(D,T+dt)$ when $f(x) = 1$, and $x \notin Capt(D,T+dt)$ when $f(x) = 0$.

As in [Rouquier *et al.*,], we use kd-trees to discover the boundary of $Capt(D,T+dt)$. The kd-tree subdivides the search space E. It focuses on the boundary by construction, so it reduces the number of calls to the oracle f, compared to the extensive search on the grid.

We first build a kd-tree g_{dt} to approximate $f_{(D)}$. So g_{dt} is an approximation of the characteristic function of $Capt(D,dt)$. We then recursively build a kd-tree g_{T+dt} that approximates $f_{g_{(T)}}$. By construction $g_{(T+dt)}$ is an approximation of the characteristic function of $Capt(D,T+dt)$.

3.2 Algorithm

The kd-tree $g = g_T$ is built by refining its root node which contains the whole space E. To this root node is associated a point x_0 of D such that an evolution starting at x_0 is still in D at time step T. Since D is a viability domain, any point in D is suitable. The root node is labeled to 1, since its associated point x_0 is in $Capt(D,T)$.

Nodes that are refined are always divided in two halves along their largest dimension, unless they meet the stopping criterion: a node is not divided any more if the length of its largest side is already h. These are called terminal nodes. With this splitting method, nodes are represented by a vector of half-open intervals (except for border nodes). Since intervals are split in the middle, they can't overlap, they are either disjoint, adjacent or included in one another. Nodes that have to be refined are:

- Border nodes (nodes that are adjacent to the boundary of the search space E) with label 1.

- Adjacent nodes with different labels (they are called critical pairs).

Two nodes are adjacent if and only if on each coordinate axis, one interval is included in (or equal to) the other, except for one coordinate where both intervals are adjacent (their closures have exactly one common point). When a node is refined, a point p is drawn at random (other method can be used) in each child node which is labeled with $f(p)$. .

The sketch of the algorithm is the following:

Algorithm 1 ***BuildG***$(root, h)$

`while` *count* $\neq 0$ `do` {
  count $\leftarrow 0$
  $A \leftarrow$ `NodeToDivide`$(root)$
  `foreach` $node_i \in A$ `do` {
    `if` $node_i$ *is not a terminal node* `then do` {
      count $\leftarrow$ *count* $+ 1$
      `Refine`$(node_i, h)$
      $A \leftarrow A \backslash node_i$ *(a node is refined only once)* }}
`return` $g = root$

`Refine`$(node, h)$ creates two children at node $node$, by splitting it along its largest dimension if its greater than h. If $node$ is represented by the vector of intervals $([a_i, b_i[)_{1,..,n}$ and is divided along j, its children will be have the same vector values except for the j^{th} coordinate which is $[a_j, \frac{(b_j-a_j)}{2}[$ for one child and $[\frac{(b_j-a_j)}{2}, b_j[$ (if $b_j \neq 1$) for the other $([\frac{(1-a_j)}{2}, 1]$ with $b_j = 1)$. A call to the oracle is made for the child that need it in order to attribute a label to the node. (Other methods may require 2 calls).

Algorithm 2 ***NodeToDivide***$(node)$

`if` *node is a leaf* `then`
  `if` $f(node) = 1$ `AND` *node is a border*
    `then return` $\{node\}$
  `else return`$(\emptyset)$
`else` { $(node_1, node_2) \leftarrow$ *node.children*
  result $\leftarrow \emptyset$
  result $\leftarrow$ *result* $\cup$ `NodeToDivide`$(node_1)$
  result $\leftarrow$ *result* $\cup$ `NodeToDivide`$(node_2)$
  result $\leftarrow$ *result* $\cup$ `PairsInNodes`$(node_1, node_2)$
  `return` *result* }

Algorithm 3 ***PairsInNodes***$(node_1, node_2)$

$node_1$ and $node_2$ are necessarily adjacent
result $\leftarrow \emptyset$
`if` *both* $node_1$ *and* $node_2$ *are leaves* `then`
  `if` $f(node_1) \neq f(node_2)$ `then`
  result $\leftarrow \{node_1, node_2\}$
`else if` *neither* $node_1$ *nor* $node_2$ *are leaves* `then`
  `foreach` $node_i$ *child of* $node_1$ `do` {
  `foreach` $node_j$ *child of* $node_2$ `do` {
    `if` $node_i$ *and* $node_j$ *are adjacent* `then`
    result $\leftarrow$ *result* $\cup$ `PairsInNodes`$(node_i, node_j)$ }}
`else do` {
  leaf $\leftarrow$ *the leaf (either $node_1$ or $node_2$)*
  node $\leftarrow$ *the other node*
  border $\leftarrow$ *axis and point of adjacency*
  label $\leftarrow 1 - f(leaf)$
  `foreach` $node_i \in$ `BoundaryNodes`*(node, border, label)* `do` {
    `if` *leaf and $node_i$ are adjacent* `then`
      result $\leftarrow$ *result* $\cup \{leaf, node_i\}$ }
`return` *result*

`BoundaryNodes`$(node, border, label)$ returns all the leaves l within $node$ such that $f(l) = label$ and that share with $node$ the same boundary $border$.

The characteristic function is built by calling `NodeToDivide` on the search space E.

3.3 Convergence

The algorithm refines a subtree (generally the root) by splitting leaves in two, so it reaches the stopping criterion in finite time.

We show here that the series of sets defined by the kd-tree algorithm converges towards the viability kernel of the associated viability problem. The method we use verifies the conditions of the convergence theorem 1 proposed in [Deffuant *et al.*, 2007]. This theorem applies when a discrete viability kernel is approximated with a classification procedure.

A general viability problem is defined by a controlled dynamics in $E \subset \mathbb{R}^p$ modeled by a set of equations such as (1), and a compact set of constraints $K \subset E$, such as (2) in which we want the dynamics to evolve:

$$\begin{cases} x'(t) & = & \varphi(x(t), u(t)) \\ u(t) & \in & U(x(t)) \subset \mathbb{R}^q \\ x(t) & \in & K \subset E \end{cases} \tag{3}$$

The viability kernel is the subset of states from which there exists a control function $u(t)$ that allows the evolution of the dynamics to stay inside K: Viab$(K) = \{x \in K \mid \exists u(.) \mid x(t) \in K \ \forall t \in [0, +\infty[\}$. Considering a time interval *dt*, the discrete dynamical system F associated to 3 is a correspondence $E \mapsto E$ which associates to a state x its set of successors. We assume that F is μ-Lipschitz, which means that the images of two vectors x and y can't diverge from more than $\mu d(x, y)$:

$$F(x) = \{x + \varphi(x, u)dt, u \in U(x)\}. \tag{4}$$

As for the language competition model, we consider that an oracle f is available, to tell whether at the following time step a state evolves towards a given set S. In order to guarantee the convergence toward the viability kernel it is necessary to consider an augmented set; The oracle is assumed to be defined from E to $\{0, 1\}$ in the following way: $\hat{f}_{(S)}(x) = 1$ if and only if there exists $u \in U(x)$ such that $d(F(x), S) \leq \mu\beta(h)$, otherwise $\hat{f}(x) = 0$ (where $\beta(h)$ depends on the grid and tends to 0 as h tends to 0). This means that the oracle $\hat{f}$ is based on the characteristic function of $S \cup (\cup_{x \in S} B(x, \mu))$. We have $\hat{f}_{(S)} = f_{S \cup (\cup_{x \in S} B(x, \mu))}$.

We want to prove that algorithm 1 converges to the characteristic function of the viability kernel $viab_F(K)$ of the discrete dynamic (4) when its stopping criterion h tends to 0 and when the viability kernel has sufficiently smooth properties.

Theorem [Deffuant *et al.*, 2007] assures that the series of sets built with a learning procedure verifying conditions (5) and (6) converges towards the viability set. In order to apply this theorem, we consider a discrete grid K_h induced by the stopping criterion h. For example, the points of the grid can be centered on each terminal leaf of a totally refined tree. We obviously have: $\forall x \in K, \exists x_h \in K_h$, such that $d(x, x_h) \leq \beta(h)$, with $\beta(h) \to 0$ when $h \to 0$. (Actually $\beta(h) = \frac{h}{2}\sqrt{p}$). In the following we omit the stopping criterion index h when it is not necessary. At each time step we define a discrete set $K^{n+1} \subset K^n \subset K_h$, and we build a kd-tree g on the sets K^{n+1}, $K_h \backslash K^{n+1}$ with algorithm 1. K^{n+1} gathers the points of K^n such that the dynamics (4) send them near $g(K^n)$ (at a distance less than $\mu\beta(h)$). We note g^n the kd-tree built with the subsets of K^n and $K_h \backslash K^n$ that correspond to points x_h for which the oracle $\hat{f}$ has been called when refining the tree. We note $G^n = \{x \in K \mid g^n(x) = 1\}$. We have $K^0 = K_h$ and $G^0 = K$. Then:

$$K^{n+1} = \left\{ x_h \in K^n \mid \hat{f}_{G^n}(x_h) = 1 \right\}.$$

Since $K^{n+1} \subset K^n \subset K_h$ it converges to a limit L_h. We note g_h the kd-tree built on the subsets L_h and $K_h \backslash L_h$, and $G_h = \{x \in K \mid g_h(x) = 1\}$.

Theorem 1 (fact) *If the learning procedure g verifies the following conditions (5) and (6), then G_h converges to Viab(K) when $h \to 0$.*

$$\exists \lambda \geq 1,\ \forall n > 0,\ (g^n(x) = 1) \Rightarrow d(x, K^n) \leq \lambda\beta(h) \quad (5)$$

$$\forall n > 0,\ (g^n(x) = 0) \Rightarrow d(x, K_h \backslash K_h^n) \leq \beta(h) \quad (6)$$

We consider that the viability kernel $viab_F(K)$ (and its complementary set in K) verify the following conditions: We note $V_\epsilon(S) = \{x \in S \mid B(x, \epsilon) \subset S\}$ (V is an open space in S)

$$\exists \epsilon > 0 \mid V_\epsilon(viab_F(K)) \text{ and } V_\epsilon(K \backslash viab_F(K)) \text{ are path-connected.} \quad (7)$$

$$\begin{cases} \forall x \in viab_F(K), & d(x, V_\epsilon(viab_F(K))) \leq \epsilon d \\ \forall x \in viab_F(K), & d(x, V_\epsilon(viab_F(K))) \leq \epsilon d \end{cases} \quad (8)$$

The path-connectivity of the viability kernel and its complementary are insured when K is simply connected and sufficiently regular. Conditions 8 insure that when the stopping criterion h is small enough, then there are no more thin tentacle that can't be seen through the grid.

Theorem 2 *The series of characteristic functions defined by the kd-trees build with algorithm 1 converges to the characteristic function of the viability kernel when the stopping criterion tends to* 0.

Proof: The learning procedure g based on algorithm 1 verifies the conditions of Theorem 1.

We first prove condition (6). Let x be a state in K such that $g^{n+1}(x) = 0$. This means that x belongs to a leaf L whith labeling point $x_h \in K \backslash K^{n+1}$. If L is a terminal node, then it is divided to the stopping criterion, and we have: $d(x, x_h) \leq \beta(h)$. If the leaf L of x is not a terminal node, let us consider the terminal node N to which x would belong if the node L was divided into a subtree to the stopping criterion. Let y_h be the labeling point of N: we have $d(x, y_h) \leq \beta(h)$. We also have $N \subset L$. Since L is not a terminal node, the closest labeling point y of a terminal node with label 1 is at least separated from L by another terminal node with label 0. So necessarily $y_h \neq y$ and so $y_h \in K^{n+1}$ and condition (6) is fulfilled. Since the kd-tree algorithm 1 proceeds in the same way whatever the label of the nodes, the condition (5) is also fulfilled. •

The oracle $\hat{f}$ is defined with the following procedure: Let g be a kd-tree build with algorithm 1. g is the characteristic function of G. We build a kd-tree $\hat{g}$ by changing the label of critical leaves to 0, then refining the tree again with algorithm 1. This adds a level of new critical leaves with label 0 (refined from larger nodes with label 0), that are adjacent to the leaves whose label have been modified. If we note $G = \{x \in K, g(x) = 1\}$ and $H = \{x \in K, \hat{g}(x) = 0\}$, we have: $h\sqrt{p} \geq d(G, H) \geq h$. This operation is repeated k times until $kh \geq \mu\beta(h)$. So the use of $\hat{g}^(k)$ is a bounded approximation of the oracle $\hat{f}$. This guarantees the convergence of the process to the viability kernel.

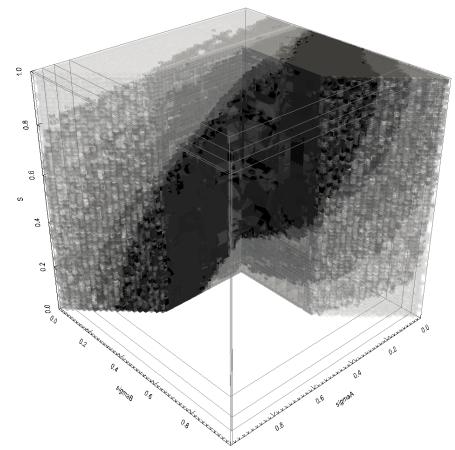


Figure 2: Level set of the resilience at 0.5; 0.2; 0.1; 0.08; 0.05; 0.025, corresponding to time level 2, 5 ,10, 15, 20, 40. Stopping criterion $h = 2^{-18}$. Lighter colors correspond to less resilient states.

4 Preservation of language diversity

4.1 Full resilience of bilingual societies

We used algorithm 1 in order to compute an approximation of the capture basin of the viability domain D of the language competition problem. [2]

From the viability domain D, the algorithm computes the successive approximations of the capture basin. Figure 2 shows the capture basin of D at different time horizons. The boundaries are the level sets of the resilience.

[2] For this experiment we use directly f as the oracle function and not $\hat{f}$, and the points associated to the leaves are drawn at random. It reduces the computation time (but the convergence is no longer strictly decreasing).

Figure 3 shows the evolution of the volume of the different approximation of the capture basin in language competition problem when increasing the time horizon. Since we necessarily have $\sigma_A + \sigma_B \leq 1$, the volume of the state space is 0.5. We can see that the volume of the capture basin actually converges to 0.5. This means that bilingual societies are fully resilient: the capture basin of the viability domain covers the entire space. So whatever the perturbations, there always exists a control policy that allows to return in the set of desirable states in less than 25 time steps, so the global resilience is $r = 0.04$.

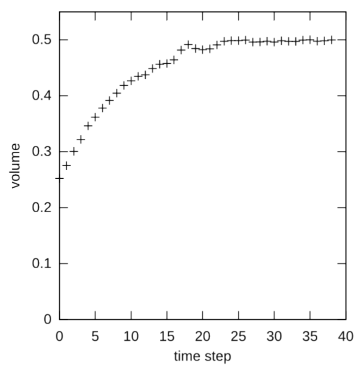


Figure 3: Evolution of the volume of the capture basin with time horizon. Convergence to 0.5 occurs around time step 25. (In this experiment the splitting direction in refined node is selected randomly, so the convergence is not monotonous)

With this information it is possible to accept temporary undesirable situations from the language diversity viewpoint, such as receiving refugees, which can shift the present state in D to a state outside D. For instance if the initial state is $(\sigma_A = 0.28, \sigma_B = 0.48, s = 0.40)$ it can be decided to accept a disturbance to the state $(\sigma_A = 0.22, \sigma_B = 0.54, s = 0.40)$ since its returning time step is only 4.

4.2 Choosing a policy after a disturbance

If a disturbance is inescapable (for instance aging of population, etc.), the computation of the capture basin provides control functions in order to return to the set of desirable states D. For example, we consider a bilingual society which state is supposed to be $(\sigma_A = 0.35, \sigma_B = 0.35, s = 0.2)$. This means that the proportion of bilingual speakers is 30%. This state is inside D, which means that an appropriate control of s can maintain the bilinguism property. We suppose now that a perturbation occurs, like a large immigration of speakers of language A. It changes the proportion of speakers of language B from 0.35 to 0.41 and the proportion of speakers of language A to 0.30. The new state $P = (0.30, 0.41, 0.2)$ is not in D. P is very close to D but even with the maximal control available $\frac{ds}{dt} = 0.1$, the dynamics drives the system outside the constraint set K (at time step 3), as it can be seen on figure 4. Since the system is resilient, we already know that it is possible to return to the set of desirable states K. The resilience of P is $r(P) = 0.14$, which means that it can return in 7 time steps to point $Q = (0.2, 0.61, 0.90)$.

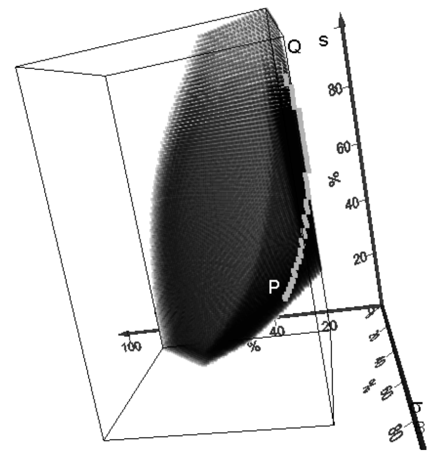


Figure 4: Resilient trajectory from the disturbed state P towards the reentry point Q. The model provides the continuous trajectory, while the kd-tree provides the control values.

5 Conclusion

We have presented in this article a new method to compute the level sets of the resilience defined in the framework of the viability theory. This method uses kd-trees with a stopping criterion h to store the characteristic function of the capture basin of the viability kernel (which provides the level set of the resilience). It uses a slightly modified kd-tree to approximate the oracle function that allow to compute these sets. The number of calls to the oracle function is limited to the number of nodes of the kd-trees. The building function concentrates these calls at points near the boundary of the viability set. We have proved that this algorithm converges towards the viability kernel when the stopping criterion h tends to 0. Moreover, the algorithm that codes the dynamics is a black box for the algorithm that build the viability kernel (or the capture basin). This modular aspect guarantees the reusability of our approach to other dynamics (the code is available as free software). We have shown how this method can be used in a language competition problem. We have shown how a resilience value can be defined. We have computed the level sets of the resilience and we have shown how it can be used to propose action policy in order to guarantee the language coexistence. This method focuses on the boundary of the viability sets rather than on the sets themselves, so it allows to consider state space with one additional dimension But it still suffers the curse of dimensionality. Further work is in progress in order to parallelize the refining part of the algorithm. Another limit is that in order to provide efficient and realistic action policies it is necessary to know the distance of a state to the boundary of the viability kernel. The algorithm can easily provide an approximation of this distance, so further work also concentrates on this point.

References

[Abrams and Strogatz, 2003] D.M. Abrams and S.H. Strogatz. Modelling the dynamics of language death. *Nature*, 424(6951):900, 2003.

[Aubin *et al.*, 2011] J.-P. Aubin, A. Bayen, and P. Saint-Pierre. *Viability Theory: New Directions*. Springer, 2011.

[Aubin, 2001] J.-P. Aubin. Viability kernels and capture basins of sets under differential inclusions. *SIAM Journal Control Optim.*, 40(3):853–881, 2001.

[Bernard and Martin, 2012] C. Bernard and S. Martin. Building strategies to ensure language coexistence in presence of bilingualism. *Applied Math. and Comp.*, 218(17):8825 – 8841, 2012.

[Bonneuil, 2006] Noel Bonneuil. Computing the viability kernel in large state dimension. *Journal of Mathematical Analysis and Applications*, 323(2):1444 – 1454, 2006.

[Bruckner *et al.*, 2003] T. Bruckner, G. Petschel-Held, M. Leimbach, and F. L. Toth. Methodological aspects of the tolerable windows approach. *Climatic Change*, 56:73–89, 2003.

[Deffuant *et al.*, 2007] G. Deffuant, L. Chapel, and S. Martin. Approximating viability kernels with support vector machines. *IEEE T. Automat. Contr.*, 52(5):933–937, 2007.

[Doyen and Saint-Pierre, 1997] L. Doyen and P. Saint-Pierre. Scale of viability and minimum time of crisis. *Set-Valued Anal.*, 5:227–246, 1997.

[Martin, 2004] S. Martin. The cost of restoration as a way of defining resilience: a viability approach applied to a model of lake eutrophication. *Ecol. Soc.*, 2(9):8, 2004.

[Perrings, 2006] C. Perrings. Resilience and sustainable development. *Environ. Devel. Econ.*, 11:417–427, 2006.

[Reuillon *et al.*, 2010] Romain Reuillon, Florent Chuffart, Mathieu Leclaire, Thierry Faure, Nicolas Dumoulin, and David Hill. Declarative task delegation in OpenMOLE. In *HPCS*, pages 55–62, 2010.

[Rouquier *et al.*,] J.-B. Rouquier, I. Alvarez, R. Reuillon, and P.-H. Wuillemin. A kd-tree algorithm to discover the boundary of a black box hypervolume. Technical Report hal-00816704, LIP6.

[Saint-Pierre, 1994] P. Saint-Pierre. Approximation of the viability kernel. *Applied Mathematics & Optimisation*, 29:187–209, 1994.

[Sicard *et al.*, 2012] M. Sicard, N. Perrot, R. Reuillon, S. Mesmoudi, I. Alvarez, and S. Martin. A viability approach to control food processes: Application to a camembert cheese ripening process. *Food Control*, 23(2):312 – 319, 2012.

Collective management of environmental commons with multiple usages: a guaranteed viability approach.

I. Alvarez[1,2], L. Zaleski[3], J-P. Briot[3], M. de A. Irving[4]

[1] UCA, INRAE, LISC
9 avenue Blaise Pascal CS 20085, F-63178 Aubière, France
e-mail: Isabelle.Alvarez@inrae.fr
[2] CNRS, ISC-PIF
113 rue Nationale, F-75013, Paris, France

[3] SU, CNRS, LIP6
4 place Jussieu, F-75005, Paris, France
e-mail: laetitia.zaleski@gmail.com, Jean-Pierre.Briot@lip6.fr

[4] EICOS, IP, UFRJ
Avenida Pasteur, 250 Rio de Janeiro 22290-240, BR
e-mail: mirving@mandic.com.br

Abstract

In this paper we address the collective management of environmental commons with multiple usages in the framework of the mathematical viability theory. We consider that the stakeholders can derive from the study of their own socioeconomic problem the variables describing their different usages of the commons and its evolution, and a representation of the desirable states for the commons. We then consider the guaranteed viability kernel, subset of the set of desirable states where it is possible to maintain the state of the commons even when its evolution is represented by several conflicting models. This approach is illustrated on a problem of lake eutrophication.

Key words viability theory; guaranteed viability; collaborative decision; environmental commons; lake eutophication

1 Introduction

Sustainable use of natural resources, environmental conservation, social inclusion and welfare, economic activity and development imply generally conflicting management objectives. In *the tragedy of the commons*, [Hardin, 1968] highlights the exhaustion of open-access resources by numerous users with similar view, but the same analysis can be done with different types of users whose activity is based on the resource. A lot of work on sustainability of natural resources is still focused on allocation problem, where stakeholders are considered as competitors for the share of quotas, for example for the regulation of fisheries or water sharing (see references for instance in [Parrachino et al., 2006], [Oubraham and Zaccour, 2018]).

In order to take into account the different types of stakeholders' interests, many efforts are done in the economic approach to assess the value of environmental and social services (see for example a framework in [de Groot, 2006]). When points of view are considered to be incommensurable, multi-criteria or viability theory approaches propose interesting alternatives. Even when stakeholders are considered as competitors for one common resource, these approaches allow to take into account more indicators than the level of renewable resource and the profit directly based on it. For example, in a quantitative work on fishing regulation [Dowling et al., 2020], 21 score functions are designed for the regulation of fishing, depending on fish biomass and parameters computed each year depending on the control scenario. Weighted stakeholders' preferences over the score functions are then optimized each year for different levels of the control variable. When stakeholders express different points of view, the viability theory (VT) approach allows to combine the different constraints placed on the system, without direct connection to the underlying profit of the related activity. For instance, in a hydro-power dam management problem [Alais et al., 2017], the main concern is maximizing the profit of the electricity provider with water control under uncertainty on water inflow and electricity price. Recreational and agricultural activities impose an additional seasonal constraint on the water level without further profit analysis. In [Wei et al., 2013], the multi-objective concern of a tourist city is studied through the linked evolution of the number of tourists, tourism infrastructure and environment quality. The different stakes are represented by constraints on the level of these variables. The VT algorithm identifies the area where it is possible to maintain the evolution of the three variables between these bounds. Viability approach has shown its potential in many other domains as stated in the review from [Oubraham and Zaccour, 2018]. In all these works, the model of the evolution of common resources or land uses, together with the impact of controls on the system (such as the total allowable catch in fishery regulation), is supposed to be consensual. It is generally taken from the literature or from previous work and parameters are calibrated from data and time series. In works cited in [Oubraham and Zaccour, 2018], the model representing the system at stake is always considered as consensual. In theoretical works, models can certainly be generic functions of variables, controls and uncertainties (it is the case in [De Lara and Martinet, 2009],[Křivan, 1991], [Křivan and Colombo, 1998], [Martinet et al., 2016]). But in their applications and in the other works cited in [Oubraham and Zaccour, 2018], when uncertainty is explicitly taken into account, it is in fact related to data and measurements used to assess parameters in the model, not to the definition of the model itself. As stated in [Martinet et al., 2016], uncertainty affects parameters such as growth rate, recruitment or mortality in dynamic population models, unknown or unpredictable events such as climate fluctuations, or externalities such as price, as in games against nature. Models are supposed to be consensual with their explicit hypotheses (which are generally discussed). Actually in [Little et al., 2007] three models are considered for larval dispersal and modelers can parameterize the system to run simulations with their own choice of model. It is motivated by the possibility of studying different species, so for the viability study only one model is parameterized.

However, the ComMod approach [Etienne, 2014] has shown that modeling the evolution of the system at stake is difficult and hardly consensual, since scientific or technical viewpoint can be considered by stakeholders as a viewpoint among others. The ComMod approach addresses this problem with serious games supported by simulation models [Barreteau et al., 2001] where stakeholders can test their hypotheses about the system evolution and the impact of actions. The process goes on, with additional research if necessary, until a consensus on the model is reached.

To take into account this discrepancy at the model level we consider here that stakeholders have their own model of the evolution of the system with the impact of controls. We consider that stakeholders are able to define constraints on the key variables of their usages of the commons, such as the number of tourists, the quality of water (for example measured in term of concentration of pollutant or bacteria), the quality of the environment (for example measured with biodiversity indicators related to the population of local species), etc. These constraints are generally seen as thresholds. We consider that the objective of the group of stakeholders is to define a set of states where the system can be maintained with appropriate controls. We use viability theory, in particular the concept of guaranteed viability set [Aubin, 1997], which is defined to take into account uncertainties (such as move by nature, see for instance [Bates and Saint-Pierre, 2018]).

The paper is organized as followed: we first describe the problem and our hypotheses, together with a reminder of viability theory. We describe individual and group viewpoint as viability problems, and show why in this context of several models it is more difficult to seek technically sound agreement. We illustrate this approach with a problem of lake eutrophication. In Section 3, we first present the perturbation embedding function, which allows us to consider each model as a perturbation of a central model. This formulation enables the definition of a guaranteed viability problem. In Section 4 we present and discuss the application to the management of the lake. We summarize the results and perspectives in the concluding section.

2 Definition of the problem

Let us consider an entity $\mathcal{A}$ (for instance a preserved area) in evolution, which is submitted to the management decision of a group of N members. The group has to first define a project for $\mathcal{A}$ as a set K of desirable states within which the state of $\mathcal{A}$ should remain with an appropriate set of actions depending on the state of $\mathcal{A}$. The group has then to find a solution to this project.

Each member $i \in \mathcal{N} := \{1, \ldots, N\}$ has a personal view and project for entity $\mathcal{A}$. Each member i considers that entity $\mathcal{A}$ is described by a vector of state variables $x_i \in \mathbb{R}^{n_i}$, and that its evolution is governed by a controlled dynamical system. Member i considers that the state of $\mathcal{A}$ should remain in a set of desirable states, $K_i \subset \mathbb{R}^{n_i}$. Member i deems that the controls which should be used depend on the state of $\mathcal{A}$. These admissible controls are defined by a set-valued map $U_i : \mathbb{R}^{n_i} \rightsquigarrow \mathbb{R}^{p_i}$, where $U_i(x_i)$ is the set of admissible controls that member i finds appropriate at state x_i.

Definition 1 (K_i, U_i) *defines member i's project for the management of $\mathcal{A}$, where K_i is the set of desirable sets and U_i the admissible control map.*

From the viability theory viewpoint, a solution to member i's project consists in finding the states of K_i from which there always exists a control function selected in U_i so that the state of $\mathcal{A}$ remains in K_i.

The objective of the group to define its project, compatible with every member's project, then to find a solution to it, compatible with everyone's solution.

We use as an illustration a problem of lake eutrophication as stated in [Carpenter et al., 1999]. Agricultural practice and other human activities can lead to lake pollution with phosphates. Phosphorus dynamics in the lake can lead to eutrophication, which negatively impacts the biodiversity of ecosystem, and causes serious annoyance to residents and tourism activities. We consider that a committee is formed to study and manage the problem. It is composed of farmers and local elected authorities.

2.1 What members share about their project

Assumption 1 *Members of the decision group share the knowledge about which state variables they consider for $\mathcal{A}$, so $\forall i \in \mathcal{N}, n_i = n$. We note $x \in \mathbb{R}^n$ the vector of state variables of $\mathcal{A}$ that is shared by each member of the group.*

In particular, when measurements of the state of $\mathcal{A}$ are possible, all members agree on the validity of the measure, so if the measure of the state of $\mathcal{A}$ at date T is $x(T)$, then $\forall i \in \mathcal{N}, x_i(T) = x(T)$.

In the lake and nearby farms problem, following [Carpenter et al., 1999] and [Martin, 2004], we consider that all members agree that the key variables to the problem are the Phosphorus input (noted L) and the Phosphorus concentration in the lake (noted P).

Assumption 2 *Members of the decision group share the knowledge about the control variables they consider for $\mathcal{A}$. We note $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^p$ the set-valued map of admissible control which associates the group's set of admissible controls with the state of $\mathcal{A}$: $\forall i \in \mathcal{N}, U_i = U$.*

Assumption 2 supposes that all group member's have reached a consensus regarding which control variables are to be considered. This can be achieved by restriction to the intersection of members' control set so that $\forall x \in \mathbb{R}^n, U(x) = \bigcap_{i \in \mathcal{N}} U_i(x)$. When the latter is empty, a negotiation should take place to build a non empty $U(x)$. We suppose here that such a negotiation has taken place.

In the lake and nearby farms problem, following [Martin, 2004], we consider that the committee agrees to the possibility of controlling the rate of variation of the Phosphorus input and to maintain this rate between boundaries, so $U = [u_{min}, u_{max}]$. This can be done by farmers controlling their fertilizer input, by the greater or lesser use of wetlands or by the use of water treatment plants ([Gajardo et al., 2017]).

With Assumption 1 and 2, it is possible to define the group project for $\mathcal{A}$.

Definition 2 *Let $K = \bigcap_{i \in \mathcal{N}} K_i$ with $K \neq \emptyset$, and $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^p$ a set-valued control map defined on K. (K, U) is a group project if all members agree that the state of $\mathcal{A}$ should remain in the set of desirable states K, using controls from the admissible map U.*

When $K \subset \mathbb{R}^n = \emptyset$, a negotiation should take place to build a non empty K. We suppose here that such a negotiation has taken place.

In the lake and nearby farms problem, everybody wants to keep the lake in an oligotrophic state, which supposes to set a concentration limit of Phophorus P_{max} in the lake (for example established from previous observations). Everybody also wants to maintain or develop the agricultural activity, which supposes to allow a minimum amount of Phophorus input L_{min} in the lake. So everybody agree to maintain the state of the lake described by (L, P) in a set of desirable states $K = [L_{min}, +\infty) \times [0, P_{max}]$.

When Assumption 1 and Assumption 2 are verified, all group members can describe the dynamics of the state of $\mathcal{A}$ as they seen it as a (possibly discrete) controlled dynamical system with the shared variables. In the case of the lake and neighboring farms, the dynamics for member i is supposed to be defined according to [Carpenter et al., 1999] and [Martin, 2004], by the equations of system 1 with the constraints on L_{min} and P_{max}.

$$S(b_i, r_i, q_i, m_i) \begin{cases} \frac{dL}{dt} & = & u \in U = [u_{min}, u_{max}] \\ \frac{dP}{dt} & = & -b_i P(t) + L(t) + r_i \frac{P(t)^{q_i}}{m_i^{q_i} + P(t)^{q_i}} \end{cases} \tag{1}$$

u_{min} and u_{max} are the maximum effort farmers and local authorities are ready to allow or to take to increase or decrease the phosphorus input. b_i is the rate of loss (due to sedimentation and outflow), r_i, q_i and m_i are parameters of the sygmoid-like (s-shaped) dynamics of Phophorus recycling in the lake, which are generally set by calibration from observations: r_i is the maximum rate of recycled Phophorus, m_i is the concentration of Phosphorus at which the recycling rate is half its maximum and q_i is a parameter of the steepness of the dynamics (see [Carpenter et al., 1999] for more details).

In the general case, we note $Sc(f,U)$ the continuous dynamical system defined by:

$$Sc(f,U)\begin{cases} x'(t) & = & f(x(t),u(t)) \\ u(t) & \in & U(x(t)) \subset \mathbb{R}^p \end{cases}. \tag{2}$$

where f is a function from $\mathbb{R}^n \times \mathbb{R}^p$ to $\mathbb{R}^n$ and U a set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^p$. Similarly, we note $Sd(f,U)$ the discrete dynamical system defined by:

$$Sd(f,U)\begin{cases} x^{k+1} & = & f(x^k,u^k) \\ u^k & \in & U(x^k) \subset \mathbb{R}^p \end{cases}. \tag{3}$$

We note $Sc_i = Sc(f_i,U)$ the dynamical system that described the evolution of $\mathcal{A}$ for member i in a continuous case. We note $Sd_i = Sd(f_i,U)$ the discrete dynamical system that described the evolution of $\mathcal{A}$ for member i in a discrete case. The function $f_i : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ associates the variations of $\mathcal{A}$ state variables for member i with the current values of the state and control variables.

We don't assume that the different stakeholders in the group share their dynamics. We consider here that they don't necessarily agree on dynamics, and that they are not compelled to make their belief public. But we assume that they agree to share this information with a trusted third party.

2.2 A reminder of the viability theory

Referring to [Aubin, 1991], we define viable evolutions and the viability kernel.

Definition 3 *An evolution of the system $Sc(f,U)$ (2) (resp. $Sd(f,U)$ (3)) is viable in K if and only if its trajectory remains in K. In the continuous case: $\forall t \in \mathbb{R}^+$ $x(t) \in K$. In the discrete case: $\forall k \in \mathbb{N}$ $x^k \in K$.*

Definition 4 *A set L is viable for the system $Sc(f,U)$ (2) (resp. $Sd(f,U)$ (3)) if for all $x \in L$ there is an evolution of the system $Sc(f,U)$ (2) (resp. $Sd(f,U)$ (3)) starting at x and viable in L.*

Definition 5 *The viability kernel associated to system $Sc(f,U)$ (2) (resp. system $Sd(f,U)$ (3)) under constraint K is the set of all states in K from which there is an evolution of $Sc(f,U)$ (resp. $Sd(f,U)$) starting at x and viable in K.*

Under some general conditions listed in appendix A, the viability kernel is a close set. In the interior of the viability kernel, all control are viable, so viable controls on the boundary show how it is possible to maintain the system in the constraint set. This information can be used to define control strategies.

Definition 6 *A control map with images restricted to viable controls only is called a viable regulation map.*

Proposition 1 *[Aubin, 1991]. If L is a viable set for the system $Sc(f,U)$ (2) (resp. $Sd(f,U)$ (3)), let $\tilde{U}$ be a viable regulation map, then L is a viable set for the system $Sc(f,\tilde{U})$ (2) (resp. $Sd(f,\tilde{U})$ (3). Moreover, for all $x \in L$, any evolution starting from x and governed by $Sc(f,\tilde{U})$ (resp. $Sd(f,\tilde{U})$ in the discrete case) is viable in L. L is called an invariant set for dynamics $Sc(f_i,\tilde{U})$ (resp. $Sd(f_i,\tilde{U})$.*

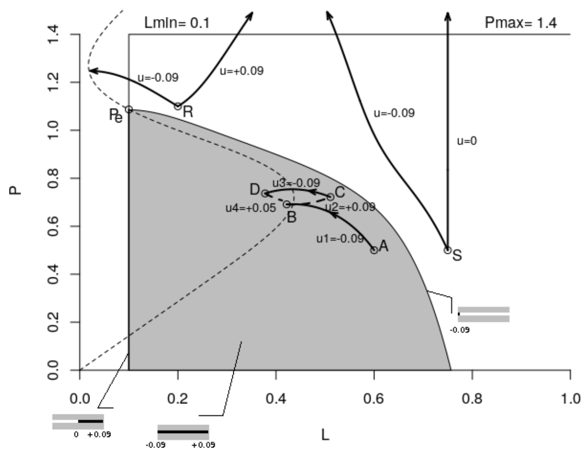


Figure 1: Viability kernel (in gray) of the lake and neighboring farms problem, with $L_{min} = 0.1$, $P_{max} = 1.4$ (L and P in μgL^{-1}), $U = [-0.9, 0.9]$, dynamics parameters value $q = 8$, $m = r = 1$, $b = 0.7$. Constraint set boundary is in black plain lines ($L = L_{min}$, $P = P_{max}$). The curve of equilibria is dashed. Viable controls are shown as a black line in cartouches. A viable trajectory starting from A is shown ($u = -0.09$ from A to B, then a cycle with $u = +0.09$ from B to C, $u = -0.09$ from C to D, $u = +0.05$ from D to B). From S where the lake is still in an oligotrophic state, even with maximum effort from the farmers ($u = u_{min}$) the concentration of Phosphorus becomes too high. From state R also outside the viability kernel, all trajectories leave the constraint set, leading the lake to eutrophic state or farmers' activity to an unsustainable state.

From any state in the viability kernel, it is always possible to find a control function that allows the state of the system to stay in the viability kernel indefinitely. Conversely, from any initial state outside the viability kernel, there is no way to prevent the exit in finite time of an evolution governed by system (2) (resp. (3) in the discrete case).

In the case of the lake and its neighboring farms, it is shown in [Martin, 2004] that the viability kernel associated to system (1) submitted to the constraint $(L, P) \in K = [L_{min}, +\infty) \times [0, P_{max}]$ is not empty when P_{max} is greater than the smallest P-value of the equilibria associated with L_{min} (an equilibrium P-value is defined by $\frac{dP}{dt} = 0$). For example in Figure 1, the state (L_{min}, P_e) is an equilibrium with $P_e \leq P_{max}$, so the viability kernel is not empty. When the curve of Equilibria intersects the half-line $(L \geq L_{min}, P = P_{max})$ at (L_e, P_{max}), the boundary of the viability kernel is delimited by the segment line $(L = L_{min}, P \leq P_{max})$, the segment line $(L_{min} \leq L \leq L_e, P = P_{max})$ and the integral curve of the dynamics with control $u = u_{min}$ arriving in (L_e, P_{max}). When the curve of Equilibria doesn't intersect the half-line $(L \geq L_{min}, P = P_{max})$, as in

Figure 1, we note P_e the P-value of the highest equilibrum on the segment line $(L = L_{min}, P \leq P_{max})$. In that case, the boundary of the viability kernel is delimited by the segment line $(L = L_{min}, P \leq P_e)$ and the integral curve of the dynamics with control $u = u_{min}$ passing through (L_{min}, P_e).

Figure 1 shows the viability kernel for the lake and neighboring farm problem in this latter case, for a given set of parameters for system (1) and constraint set K. From any state in this viability kernel, it is possible to find a trajectory that stays in the viability kernel indefinitely. Figure 1 presents an example of viable trajectory from a state in the viability kernel. It also shows examples of states outside the viability kernel; even the most severe control cannot prevent trajectories from leaving the constraint set. Either the lake will shift to an eutrophic state, or the economic activity will be jeopardized.

From a state outside the viability kernel, every evolution governed by system (1) with this particular set of parameters, choice of constraint set and control interval will exit the constraint set. In general, dealing with state outside the viability kernel implies to study the resilience (as in [Martin, 2004]) or to redefine the problem. This can be done by relaxing the constraints on the desirable set (when it is possible), by allowing more efficient control which are not presently part of the admissible controls, or by modifying the dynamics. This latter option is generally more difficult to implement, since it implies to modify the lake itself (see [Liu et al., 2015] for example of such actions).

2.3 Viewpoints as viability problems

Objectives. We assume that Assumptions 1 and 2 are verified, and that the group has defined (K, U) as its project for $\mathcal{A}$ according to Definition 2. $K \neq \emptyset \subset \mathbb{R}^n$ is the set of desirable states for $\mathcal{A}$ and $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^p$ the set-valued map of admissible controls. With this definition of the group project, each member can work on a solution according to the dynamics he assumes for $\mathcal{A}$. Finally, the objective of the group is to design a solution from all member's solutions. Figure 2 summarizes the implications of considering different usages and stakeholders for the lake and nearby farms problem. Although the intuition is to work from the set of individual solutions, in this section we show that this approach is difficult to implement.

2.3.1 Individual Viewpoint

We consider here that each member i is able to describe the evolution of the state of $\mathcal{A}$ with a controlled dynamical system. It is either a continuous system $Sc_i = Sc(f_i, U)$ (equation 2), or a discrete system $Sd_i = Sd(f_i, U)$ (equation 3). We also consider in the following that the conditions for Proposition 3 (resp. 4) are fulfilled: the viability kernel associated to member i's system and constraints is closed.

Let $L_i \subset K$ be a non-empty viable set for the continuous (resp. discrete) system $Sc(f_i, U)$ (2) (resp. $Sd(f_i, U)$ (3)) submitted to constraints K. Then from all states in L_i there is at least one viable evolution governed by $Sc(f_i, U)$

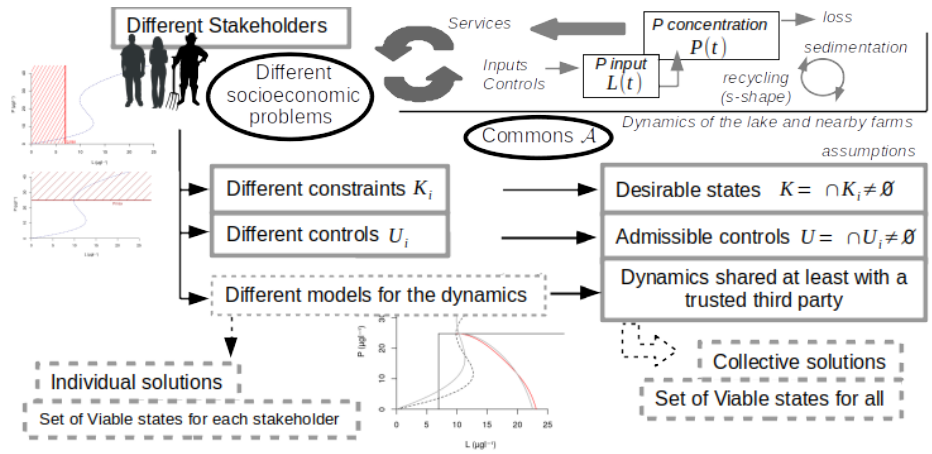


Figure 2: Diagram of the finding of management solutions for the lake and nearby farms (system $\mathcal{A}$) in the framework of viability theory with different stakeholders. Gray arrows denote relationship in the dynamics model. Large gray arrows represents interaction with stakeholders models and dynamics (which are not explicit). Black arrows represents the viability analysis process. Dotted lines and arrows show the main focus of the article.

(resp. $Sd(f_i, U)$) that stays in L_i. From member i viewpoint, L_i is a solution state set to the management of $\mathcal{A}$.

Definition 7 *$L_i \subset K$ is a solution state set for Member i for project (K, U) if L_i is a non-empty viable set for Member i's dynamics.*

We note $\text{viab}_i(K)$ the viability kernel associated to member i's project with dynamical system $Sc(f_i, U)$ (2) (resp. $Sd(f_i, U)$ (3)) submitted to the viability constraint K.

In the case of the lake and its neighboring farms, Figure 1 shows the viability kernel for the dynamics (1) submitted to constraint set $K = [L_{min}, +\infty) \times [0, P_{max}]$ for the particular values of the dynamics parameters (noted as farmers representative in Figure 3).

2.3.2 Group Viewpoint

In the following we suppose that the group project for $\mathcal{A}$ is (K, U) and that all group members can propose their own individual solution to the management of $\mathcal{A}$:

$$\forall i \in \mathcal{N}, \text{viab}_i(K) \neq \emptyset$$

We note $H = \bigcap_{i \in \mathcal{N}} \text{viab}_i(K)$. If $H = \emptyset$ a negotiation should obviously take place between stakeholders, since there is no way to operate $\mathcal{A}$ and satisfy the group members. When the intersection is not empty, it seems a good candidate. The intersection of viability kernels has already been proposed as a solution to insure the viability of two fishing fleets operating on the same resource. In [Sanogo et al., 2012], the intersection of the viability kernel of both fleets is viable for each fleet if they change their effort at the same time when necessary, which suppose a high level of cooperation. But unfortunately, it is not always the case.

Proposition 2 *The intersection of all members' viability kernels is not necessary viable for all members.*

Proof. The problem of the lake and neighboring farms gives a counter-example. We consider here two stakeholders, say a mayor and a farmer's union representative (respectively noted with m and f indices). Both stakeholders interpret the observations in different ways, so they adopt different values for the parameters of dynamics (1). Their respective viability kernel (viab_m and viab_f) associated to the constraint set $K = [L_{min}, +\infty) \times [0, P_{max}]$ are shown on Figure 3. For the particular parameters chosen, the intersection H is not empty. For $x = (L, P) \in viab_i$, viable controls are defined by $\tilde{U}_i$ with $\tilde{U}_i(L_{min}, P) = [0, u_{max}]$, $\tilde{U}_i(L, P) = \{u_{min}\}$ when (L, P) is on the boundary of viab_i with $L \neq L_{min}$, and otherwise $\tilde{U}_i(x) = U$. Nevertheless, state A in the intersection is not viable for the mayor. State A is on the boundary of the mayor's viability kernel, so $\tilde{U}_m(A) = \{u_{min}\}$ and the only viable control for the mayor is $u = -0.09$. But the trajectory starting at A and governed by (S_m) with $u = -0.09$ stays on the boundary of viab_m so it leaves H. ■

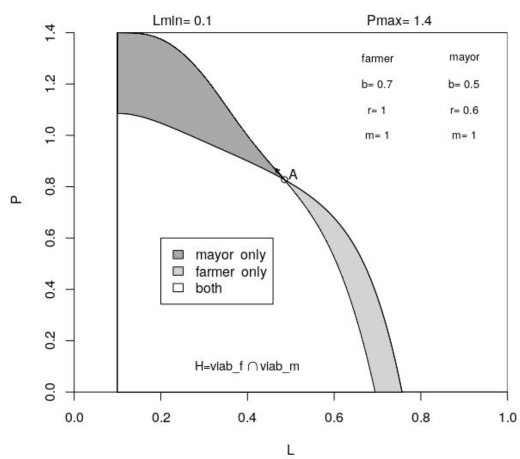


Figure 3: Viability kernels of two stakeholders in the lake and neighboring farms problem, with $L_{min} = 0.1$ and $P_{max} = 1.4$ (L and P in μgL^{-1}), $U = [-0.9, 0.9]$, shared parameters value $q = 8, m = 1$. In white, the intersection H of the viability kernels. In dark (resp. light) gray, the complementary area of the mayor (resp. farmer) viability kernel. State A is not viable in the intersection for the mayor. The arrow shows the trajectory of state A according to the mayor: it leaves the white area.

Definition 8 *Let $L \subset K$ and let $(U_i)_{i \in \mathcal{N}}$ be control maps defined on L. Let U be defined for all $x \in L$ by $U(x) = \bigcap_{i \in \mathcal{N}} U_i(x)$. U is called the intersection of $(U_i)_{i \in \mathcal{N}}$ on L.*

We note $\tilde{U}$ the regulation map defined on the intersection H of all viability kernels of the group members by the intersection of all the corresponding viable regulation map: $\tilde{U}(x) = \bigcap_{i \in \mathcal{N}} \tilde{U}_i(x)$. Obviously, if there is a state $z \in H$ such that $\tilde{U}(z) = \emptyset$, it means that members cannot agree on a way to control the evolution of $\mathcal{A}$ at this particular state. Unfortunately, even if all members agree on controls on H, it is not sufficient to reach a consensus.

Corollary 1 *Let $\tilde{U}$ be the intersection on $H = \bigcap_{i \in \mathcal{N}} \text{viab}_i(K)$ of the viable regulation map U_i on each $\text{viab}_i(K)$ of each member $i \in \mathcal{N}$. $Dom(\tilde{U}) = H$ is not a sufficient condition for H being a viable set for all members.*

Proof. In the previous example, we can derive that $\tilde{U}_m(L,P) = \tilde{U}_f(L,P)$ for all (L,P) in the intersection except on the set H_b of the boundary of H wherever $L \neq L_{min}$. On the part of the boundary of H which is the boundary of $viab_m$ only, $\tilde{U_m}(L,P) = \{u_{min}\}$, while $\tilde{U_f}(L,P) = U$ (and conversely on the boundary of $viab_f$ only). So for $(L,P) \in H_b$, $\tilde{U}(L,P) = \{u_{min}\}$ so $Dom(\tilde{U}) = H$. Nevertheless state A is not viable in H for the mayor. ■

From Proposition 2, and Corollary 1 we propose the following definition for a technically sound consensus solution to the management of $\mathcal{A}$.

Definition 9 *Let (K,U) be the projet of the management group for $\mathcal{A}$. A set of state $H \subset K$ is a consensus solution if H is a viable set for the each member and if the domain of the intersection $\tilde{U}$ of the viable regulation maps of each member on H, $(U_i)_{i \in \mathcal{N}}$, is such that $Dom(\tilde{U}) = H$.*

Actually, to be a consensus solution, a subset H of the constraint set has to be viable for all members and each viable state has to share at least one viable control for all members. In that case it is possible for the group member to reach an agreement on the control, regardless of trajectories. For example, in the discrete case, from any state x_0 of H, all group members share at least one viable control value that allows the state of $\mathcal{A}$ to stay in H. Since the dynamics they consider are different, there is generally no consensus on state x_1. But as long as the group members still share a viable control value they still can agree on it. When it is no longer the case, for example at step n, the true value of the state of $\mathcal{A}$ can be measured to continue this process from x_n as new starting point. In the continuous case, when H is a close set, such situations arise only on the boundary of H.

Figure 4 shows a consensus state set for the lake and neighboring farms system (1) with parameters of Figure 3. The associated regulation map $\tilde{U}$ is such that $\tilde{U}(L_{min}, P) = [0, u_{max}]$, $\tilde{U}(L,P) = \{u_{min}\}$ when (L,P) is on the boundary with $L \neq L_{min}$, and otherwise $\tilde{U}(L,P) = [u_{min}, u_{max}]$. For every state (L_0, P_0) of the consensus state set G, there is an evolution governed by system (1) for each stakeholder, starting at (L_0, P_0), with the same $u(0) \in$

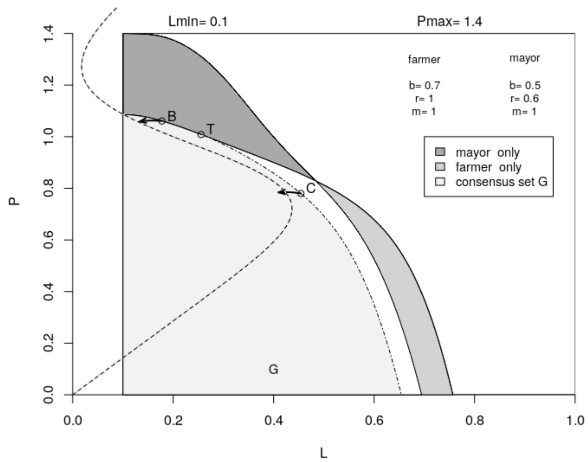


Figure 4: A consensus state set for the two stakeholders in the lake and neighboring farms problem with parameters from Figure 3. In very light gray, the consensus state set G, delimited in dot dashed line by the trajectory following the mayor's dynamics that is tangent to the boundary of the viability kernel of the farmer at state T (in plain line). From each state (L, P) of this trajectory before the tangent state T (with $L > L_T$), the evolution governed by the farmer's system starting at these states with $u = u_{min}$ leaves the boundary to evolve inside G, as it is shown for state C. Respectively, from each state of the boundary of the viability kernel of the farmer with $L_{min} < L < L_T$, the evolution governed by the mayor's system starting at these states with $u = u_{min}$ leaves the boundary to evolve inside G, as it is shown for state B. In dashed line, the line of equilibrium for the farmers representative's dynamics.

$\tilde{U}(L_0, P_0)$ that stays in G. In the interior of the intersection of the viability kernels this property is also verified since for every states in the interior all controls are viable in the viability kernel of each stakeholder. For states on the boundary with $L = L_{min}$, for both the mayor and the farmers' representative several evolutions are viable, in particular with $u = 0$. The consensus state space is delimited by the boundary of the farmers'representative and the trajectory governed by the mayor's dynamics with the minimum control value that stays in the viability kernel of the farmers'representative with largest L when $P = 0$.

In the case of the lake and neighboring farms problem, with only two group members, it is possible to define a consensus state set because of the properties of the dynamics, for which the line of equilibria is known, and the viability kernels and the trajectories corresponding to minimum control value u_{min} can be easily defined and computed [Martin, 2004]).

For more general cases it is necessary to propose a method that can be applied without such knowledge. We present such a method in the next section.

3 Consensus with guaranteed Viability

3.1 Embedding function for the dynamics

Since the group members have their own definition for the dynamics, all members can see others' definitions as perturbations of their own. We show here that is possible to define the dynamics of $\mathcal{A}$ embedding all members' definitions seen as perturbations. The dynamics of $\mathcal{A}$ depends on the state of $\mathcal{A}$, $x(t)$, on the control chosen in $U(x(t))$ and on perturbations occurring from a set $V(x(t)) \subset \mathbb{R}^q$ that depends on the state of $\mathcal{A}$. In the continuous case we have:

$$Svc(f,U,V) \left\{ \begin{array}{lll} x'(t) & = & f(x(t),u(t),v(t)) \\ u(t) & \in & U(x(t)) \\ v(t) & \in & V(x(t)) \end{array} \right. \tag{4}$$

In the discrete case:

$$Svd(f,U,V) \left\{ \begin{array}{lll} x^{k+1} & = & f(x^k,u^k,v^k) \\ u^k & \in & U(x^k) \subset \mathbb{R}^p \\ v^k & \in & V(x^k) \subset \mathbb{R}^q \ , \end{array} \right. \tag{5}$$

where f associates the new state of $\mathcal{A}$ with its present state, a control chosen in $U(x(t))$ and a perturbation in $V(x(t))$. $Scv(f,U,V)$ and $Sdv(f,U,V)$ are called dynamical controlled tychastic systems [Aubin, 1997].

Definition 10 *We say that System $Svc(f,U,V)$ (4) (resp. $Svd(f,U,V)$ (5)) embeds System $Sc(f_i,U)$(2) (resp. $Sd(f_i,U)$ (3)) for $i \in \mathcal{N}$, and call the corresponding pair (f,V) an embedding solution if and only if:*

$$\forall x \in K, \forall u \in U(x), \forall i \in \mathcal{N}, \exists v_{i,u,x} \in V(x), f_i(x,u) = f(x,u,v_{i,u,x}) \tag{6}$$

We show in appendix B that under some general conditions a System (4) (resp. 5 in the discrete case) can embed $Sc(f_i,U)$(2) (resp. $Sd(f_i,U)$ (3)) for all $i \in \mathcal{N}$.

For example, for the problem of the lake and neighboring farms, the dynamics for every group member are defined from S_{lake} in equation (1) by $f_i : \mathbb{R}^2 \times \mathbb{R} \rightsquigarrow \mathbb{R}^2$, with:

$$f_i((x_1,x_2),u) = \left(\begin{array}{l} u \\ -b_i x_2 + x_1 + r_i \frac{x_2^q}{m+x_2^q} \end{array} \right) \tag{7}$$

where parameters m and q have consensus values among the group, whether parameters b_i and r_i have not. Then, by defining $V = [\min_{i\in\mathcal{N}}(b_i), \max_{i\in\mathcal{N}}(b_i)] \times [\min_{i\in\mathcal{N}}(r_i), \max_{i\in\mathcal{N}}(r_i)]$ and f as:

$$f((x_1,x_2),u,v) = \left(\begin{array}{l} u \\ v_1 x_2 + x_1 + v_2 \frac{x_2^q}{m+x_2^q} \end{array} \right) \tag{8}$$

with $v = (v_1, v_2) \in V$, equation (6) is verified, since in this simple case we have:

$$\forall x \in K, \forall u \in U(x), \forall i \in \mathcal{N}, v_{i,u,x} = (b_i, r_i).$$

In the following, we assume that the group has defined a map f and a perturbation map V such that System (2) (resp. (3) in the discrete case) describes the dynamics of $\mathcal{A}$, embedding the viewpoint of all group members.

The objective of the group is then to find a consensus solution for $\mathcal{A}$ which will guarantee the viability for each member with shared viable controls.

3.2 Guaranteed Viability with embedding dynamics

We recall here some definitions and properties of the mathematical theory of viability, from [Aubin, 1991] and [Lavallée, 2020], relatively to guaranteed viability.

Definition 11 *A solution $x(.)$ of system (4) (resp.5) is an evolution $(t \mapsto x(t))$ (resp. $(x^k)_{k\in\mathbb{N}}$) such that there is a control function $(t \mapsto u(t))$ (resp. $(u^k)_{k\in\mathbb{N}}$) and a perturbation function $(t \mapsto v(t))$ (resp. $(v^k)_{k\in\mathbb{N}}$) such that system (4) (resp.5) is verified for almost all $t \geq \mathbb{R}^+$ (resp. for all $k \in \mathbb{N}$).*

Definition 12 *An evolution $x(.)$ (resp. (x^k)) solution of system (4) (resp.5) is viable in L if and only if its trajectory remains in L.*

Following [Aubin, 1991], [Doyen, 2000] and [Lavallée, 2020], we recall the property of guaranteed viability.

Definition 13 *(From [Aubin, 1997]) A set L verifies the property of guaranteed viability for $Svc(f,U,V)$ (4) (resp. $Svd(f,U,V)$ (5)) if there is a regulation map $\tilde{U}$ defined on L with non empty subset of U images, i.e. $\forall x \in L, \tilde{U}(x) \neq \emptyset$ and $\tilde{U}(x) \subset U(x)$ such that for all x_0 in L, all evolutions starting at x_0 and governed by $Svc(f,\tilde{U},V)$ (resp. $Svd(f,\tilde{U},V)$) are viable in L.*

Definition 14 *The guaranteed viability kernel associated to a set K is the largest set in K with property of guaranteed viability (for λ-Lipschitz controls in the continuous case - see appendix A for definition).*

We note $Guar_{Sdv(f,U,V)}$ the guaranteed viability kernel associated to a set K for the discrete dynamics (5). In the continuous case (4), we note it $Guar_{\lambda,Scv(f,U,V)}$.

We have seen that the intersection of each member's solution is not necessarily a solution for all members. We are going to show that the guaranteed viability kernel is a consensus solution (as in Definition 9).

Let $Scv(f,U,V)$ (resp. $Sdv(f,U,V)$) be an embedding solution for all group members, which fulfilled conditions of Proposition 5. Let $L \neq \emptyset$ be the guaranteed viability kernel for system $Scv(f,U,V)$ with λ Lipschitz constant (resp. $Sdv(f,U,V)$) associated to constraint set K. Then we have the following property:

Theorem 1 *The guaranteed viability kernel associated to K for $Scv(f,U,V)$ (with λ Lipschitz constant in the continuous case) (resp. $Sdv(f,U,V)$ in the discrete case) is a consensus solution to the management of $\mathcal{A}$.*

The demonstration can be found in appendix A.3. We note Guar_K the guaranteed viability kernel and $\tilde{U}$ the associated viable regulation map. The basic idea is that in Guar_K, from member i's perspective, an evolution governed by system $Sc'_i = Sc(f_i, \tilde{U})$ (resp. $Sd'_i = Sd(f_i, \tilde{U})$ in the discrete case) is also governed by the embedding system $Scv(f, U, V)$ (resp. $Sdv(f, U, V)$), so it remains in Guar_K, therefore Guar_K is a viable set for each member i. ■

Under some general conditions, the guaranteed viability kernel is a close set (see Proposition 5 in appendix A.3). If the guaranteed viability kernel is closed, it is possible to retrieve the value of viable controls on its boundary. This information could be used to anticipate or to design control strategies that keep an evolution away from the boundary.

4 Application to the problem of lake eutrophication

4.1 Lake Bourget case

We consider Lake Bourget, which is the biggest lake located entirely within France. It is monitored by the inter-syndicate committee CISALP, which is in charge of the design, animation and management of contractual actions for depollution and restoration of Lake Bourget. The lake had experienced a long eutrophication period, since in 1974, incoming amount of P in the lake was around 300 tons per year, in 1989 the in-lake concentration was above 150 $mg.m^{-3}$ ([Vinçon-Leite and Tassin, 1990]) where OECD norms assess the in-lake concentration to a maximum of $P_o = 10$ $mg.m^{-3}$ (equivalent of 36 tons) for the oligotrophic state (from [Vollenweider, 1982]). Similarly a threshold for mesotrophic state $P_m = 35$ $mg.m^{-3}$ can be defined. With concentration above P_m the lake is supposed to be in an eutrophic state. Since the lake is monitored and the data are available, it is possible to calibrate the equations of system 1 for lake Bourget. For Phosphorus unit in $mg.m^{-3}$ states values and parameters r and m are divided by the volume of the lake in billions of m^3. The volume of the lake is $v = 3.6\ 10^9\ m^3$ and supposed to be constant. Calibration coefficients from [Brias et al., 2018] are given in Table 1.

Lake Bourget offers multiple services apart from being a freshwater reserve. It is an area of major ecological interest for flora, fauna and the diversity of its biotopes. Several areas of the lake are classified as protected area. It supports a

Table 1: Parameters of the Lake Bourget model

Parameter	b	r	q	m
	state unit in tons			
Value	2.2676	367.04	2.222	96.85
	state unit in $mg.m^{-3}$ (or $\mu g.l^{-1}$)			
	2.2676	101.96	2.222	26.90

lot of tourism and recreational activities (water-sport, fishing, beaches, marina) and cultural activities linked in particular to historical heritage and literature. Several other services are being considered, such as the production of hydrothermal energy. Although the state of the lake has considerably improved, it is still considered as oligo-mesotrophic. Its dynamics can be unstable due to P loading and several blooms of cyanobacteria have been observed lately.

Agriculture is now the main source of P loading since major prevention measures have been taken since 1980. In particular the effluents of water treatment plants are no longer rejected in the lake.

The control of incoming P is considered as essential because of the potential lagged impact of P release from sediments ([Jacquet, 2018]).

We consider a scenario where the CISALP in its mission of negotiation would approach agricultural unions, local representatives and managers of tourism activities, to form a committee in order to control P loading in the lake to prevent eutrophication and its consequences. The effort on P loading we consider is a limit on its variation as it is common in environmental actions. It can be implemented by changes in agricultural practice and by the use of wetlands or retention basins. Actually retention basins are being built to regulate the incoming of polluted water.

As in Section 2, we consider that the committee members are aware of models regarding lake Bourget (such as [Brias et al., 2018]), but they can disagree on the value of parameters or even on the model formulation. We consider that they can agree on a set K of desirable states and on a set of admissible control U, or at least they can consider several scenarios for the definition of theses sets.

4.2 Scenarios and Results

We consider a scenario where members of the committee agree on the state variables and the possibility of controlling the rate of P loading (L). They consider different parameters sets for model (1), and possibly a different formulation for the process of recycling from sediments. Some members consider that the recycling process can have more effect at low value of P total than with model (1). A different formula for the sigmoid-like function is used in that case as shown in Equation (9), with parameter λ_i controlling the shape instead of q_i in model (1). For small value of $\lambda_i > 0$ the recycling occurs also for low level of in-lake P, so the lower branch of the equilibrium curve is actually higher.

$$S'_i \left\{ \begin{array}{lcl} \frac{dL}{dt} & = & u \in U = [u_{min}, u_{max}] \\ \frac{dP}{dt} & = & -b_i P(t) + L(t) + r_i \frac{P(t)}{P(t) + m_i e^{(-\lambda_i (P(t) - m_i))}} \end{array} \right. \tag{9}$$

It is possible to embed both model types by considering an additional parameter $\alpha_i \in [0, 1]$ which controls the predominance of one type over the other. The corresponding model is represented in Equation (10).

$$B_i \begin{cases} \frac{dL}{dt} = u \in U = [u_{min}, u_{max}] \\ \frac{dP}{dt} = -b_i P(t) + L(t) + (1-\alpha_i) r_i \frac{P(t)^{q_i}}{m_i^{q_i} + P(t)^{q_i}} + \alpha_i r_i \frac{P(t)}{P(t) + m_i e^{(-\lambda_i (P(t) - m_i))}} \end{cases} \quad (10)$$

The committee members' believes we consider are summarized in Table 2. Regarding the definition of the constraint set, we consider that agricultural activity leads to at least 25 tons of incoming P each year. This value is arbitrary but it is lower than the mean loading between 2004 and 2016 which was above 33 tons/year (see [Brias et al., 2018]). So we choose as lower limit $L_{min} = 25/v \approx 6.94\ \mu g.l^{-1}$. Considering the desirable threshold for in-lake P, we consider an optimistic scenario with the value of the mesotropic equilibrium as maximum, $P_{max} = 24.76\ \mu g.l^{-1}$. The constraint set for this scenario is $K = \{(L,P), L \geq L_{min}, P \leq P_1\}$ (respectively P_2 for K_2). As possibility of control, we consider that the maximum rate for the reduction of incoming P is half the maximum difference Δ of loading between two consecutive years between 2004 and 2016. For the increase of the loading we consider that the maximum rate can be Δ. So the set of admissible control is $U = [-\frac{\Delta}{2}, \Delta]$, with $\Delta \approx 3.15\ \mu g.l^{-1}y^{-1}$. When parameters are in a range, we consider the embedding dynamics Sv as in equation 8 with the corresponding parameter as v and V its range. For each member i it is possible to define a viability problem either as in Section 2.3.1, when parameters have fixed values, or a guarantied viability problem as in Section 3.2, when parameters are in a range. We note viab$_i$ the viability kernel associated to Member i's project (K, U) and dynamics S_i (or S'_i) with fixed parameters (or Sv_i or S'_i with parameter in a range). Figure 5 shows the viability and guarantied viability kernels computed for scenario 1 for members 2 to 4.

Applying the method described in the previous section, we define an embedding function f_B for the group from model ()9) and Table 1. We then define

Parameters b, r, m in $\mu g.l^{-1}$	b_i P loss	r_i max. rate	m_i P value at half max. rate	α_i model type	q_i steepness model (1) S_i	λ_i steepness model (9) S'_i
Table 1 / Member 1	2.2676	101.96	26.90	1	2.222	-
Member 2	2.2676	101.96	26.90	1	$[2.2, 2.3]$	-
Member 3	$[2.2, 2.3]$	101.96	26.90	1	2.222	-
Member 4	2.2676	101.96	26.90	0	-	$[1/19, 1/16]$

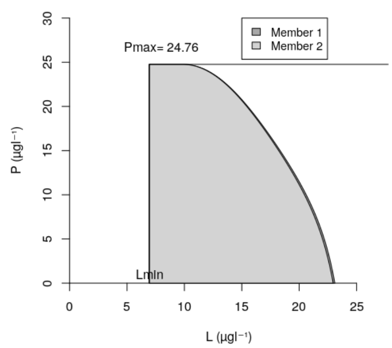
Table 2: Believes regarding the model and parameters of Lake Bourget dynamics.

the guarantied viability problem Bv (equation 13) associated to f_B.

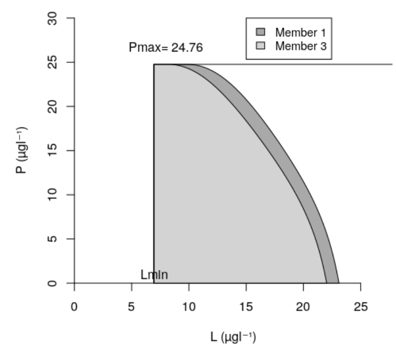
$$f_B(x,u,v) = \begin{pmatrix} u \\ -v_1 P + L + r\left((1-v_2)\frac{P^{v_3}}{m^{v_3}+P^{v_3}} + v_2 \frac{P}{P+me^{(-v_4(P-m))}}\right) \end{pmatrix} \tag{11}$$

where v_1 stands for parameter b_i, v_2 for α_i, v_3 for q_i and v_4 for λ_i, with :

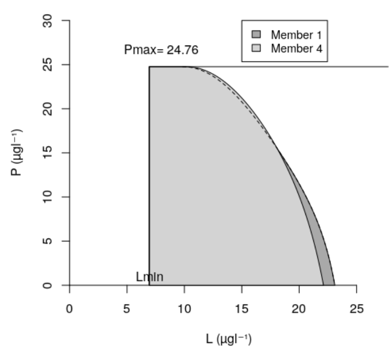
$$\begin{cases} x &= (L,P) \in K \\ u &\in U = [-\frac{\Delta}{2}, \Delta] \\ v &\in V = [2.2, 2.3] \times [0,1] \times [2.2, 2.3] \times [1/19, 1/18] \end{cases} \tag{12}$$



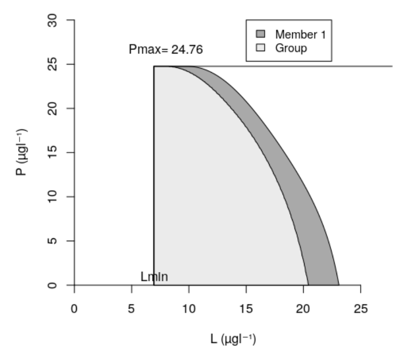
(a) (a) Member 2 versus Member 1.



(b) Member 3 versus Member 1.



(c) Member 4 guarantied viability kernel versus Member 1.



(d) Guarantied Viability kernel $Guar_{f_{Bd}}$ associated to the group versus Member 1.

Figure 5: Guarantied viability kernel for each member and the group versus Member 1's viability kernel. Computation with R ([Team, 2010]) and ViabLab ([Désilles, 2020])

.

$$Bv\begin{cases}(L,P)'(t) & = & f_B((L,P)(t),u(t),v(t))\\ u(t) & \in & U\\ v(t) & \in & V\\ (L,P)(t) & \in & K\end{cases} \tag{13}$$

Since U and V are constant function of (L,P), and since f_B is Lipschitz, the conditions of Proposition 5 are fulfilled. Since U is constant, it is Lipschitz for every $\lambda > 0$, so the guarantied viability kernel $Guar_{\lambda,f_B}(K)$ associated to problem (13) is closed and has the property of guarantied viability. It is then from Theorem 1 a consensus set of states for the four member of the committee whose believes are summarized in Table 2. To compute an approximation of $Guar_{\lambda,f_B}(K)$, we use the ViabLab library [Désilles, 2020], developed by A. Désilles and used in [Durand et al., 2017]. This library uses the convergence conditions of the algorithm established by P. Saint-Pierre [Saint-Pierre, 1994]. Since the ViabLab library requires presently for the computation of guarantied viability kernel discrete problems in time and space, we defined a discretized version of the viability problem, with function f_{Bd} from equation (14), with a discretization parameter $\tau = 0.1$ for which the dynamics are stable. We also used projection on grid method from [Lavallée, 2020] to minimize discretization error.

$$f_{Bd}((L,P),u,v) = \begin{pmatrix} L+\tau u \\ P+\tau\left[-v_1P+L+ \right. \\ \qquad \left. r\left((1-v_2)\frac{P^{v_3}}{m^{v_3}+P^{v_3}}+v_2\frac{P}{x_2+me^{(-v_4(P-m))}}\right)\right]\end{pmatrix} \tag{14}$$

The resulting approximation $Guar_{f_{Bd}}(K)$ is shown on Figure 5d. The guarantied viability kernel for the group is a viable set for all members, and the viable controls on its boundary are viable controls for all members.

4.3 Discussion

Depending on the dynamics and the beliefs of the different group members, the guarantied viability kernel computed following the approach of Section 3.1 could be smaller than the largest viable set corresponding to the union of the parameters set of each group member. For instance, with $(b,r) \in \{(2.1,100),(2.2,80)\}$, it is possible to design a viable set for these two values only, solving the problem with the computation of integral curves as it is done in Section 2.3 for the lake problem (see figure 4). Whereas following the method in Section 3.1, in order to respect the conditions of VT theorems and use the viablab library it is necessary to define a guaranty viability problem for $(b,r) \in [2.1,2.2]\times[80,100]$. But when the dimension of the state space is greater than 2, the first method is virtually impossible to implement with a generic module (since a specific mathematical study of the dynamics is necessary).

The viability algorithm is exponential with the dimension of the space in the general case, so it can be very slow, in particular when the viability kernel is empty or with high dimension problems. For each alternative model definition a tyche has to be considered, which increases the dimension of V linearly with the number of models. The computation of $Guar_{f_{Bd}}(K)$ on Figure 5d takes 938.64s with a processor i7-8650U CPU @ 1.90GHz $\times$ 8 and 15.5GiB RAM, with an accuracy of 1000 points/axis and a discretization step of 11 for control and the model type and 5 for the three other tyches (b,q,λ). It takes 1243.86 s for a scenario with $P_{max} = 15.0$ $\mu g.l^{-1}$ (all parameters being equal), and the guarantied viability kernel is not empty. Increasing the dimension of V can also lead to much longer computation time. For instance, it takes 7122.99 s for the scenario with $P_{max} = 15.0$ $\mu g.l^{-1}$, all parameters being equal except for $m \in [26.0, 27.0]$ as additional tyche. In that case the guarantied viability kernel is empty. This method has been used for a dimension 3 problem of management of marine protected area [Zaleski, 2020]. With an accuracy of 100 points/axis and dimension 2 controls and tyches (with discretization step of 11), the computation time is 437.94 s. It is 3153.31 s with an accuracy of 300 points/axis and 5 steps for each tyche.

Regarding the model of Lake Bourget itself, we consider here a single control for different practices (use of wetlands, use of retention basins, different farmer practices), and the value of its range is consistent with observations but arbitrary. The model could be improved by taking into account more detailed mechanisms for these different types of control and their relation to soil leaching and rainfall. The lake is considered as homogeneous, which seams a reasonable assumption regarding the water resident time of 14 years ([Brias et al., 2018]). On the other hand, since blooms of cyanobacteria are often localized, it could be useful to use spatial and weekly data to assess the size of perturbations and take them into account to consider robustness issues as defined in [Martin and Alvarez, 2019].

5 Conclusion

In this paper, we have presented a method for reaching a viability-based consensus for the management of commons with multiple usages. We have proposed a definition of the management project and stakeholder viewpoints, and a viability-based definition for the consensus solution as a viable set for all stakeholders with conditions on their regulation map. We have defined embedding functions that allow to compute a guarantied viability kernel for the associated dynamics. We have then shown that this guarantied viability kernel is a consensus solution (Theorem 1). It can then be computed with algorithms used for viability kernel approximation. We have then applied this method to a management scenario for Lake Bourget. The main interest of this method is that stakeholders can retain their vision of the dynamics. Negotiations can focus on the definition of desirable states and admissible actions. It also prepare the way for an alternative to agent based modeling when dealing with stakeholders for management of commons.

References

[Alais et al., 2017] Alais, J.-C., Carpentier, P., and De Lara, M. (2017). Multi-usage hydropower single dam management: chance-constrained optimization and stochastic viability. *Energy Systems*, 8(1):7–30.

[Aubin, 1991] Aubin, J.-P. (1991). *Viability Theory*. Birkhäuser, Basel.

[Aubin, 1997] Aubin, J.-P. (1997). *Dynamic economic theory: a viability approach*, volume 5. Springer Verlag.

[Barreteau et al., 2001] Barreteau, O., Bousquet, F., and Attonaty, J.-M. (2001). Role-playing games for opening the black box of multi-agent systems: method and lessons of its application to Senegal River. *Journal of Artificial Societies and Social Simulation*, 4(2):5.

[Bates and Saint-Pierre, 2018] Bates, S. and Saint-Pierre, P. (2018). Adaptive policy framework through the lens of the viability theory: A theoretical contribution to sustainability in the anthropocene era. *Ecological Economics*, 145:244–262.

[Brias et al., 2018] Brias, A., Mathias, J.-D., and Deffuant, G. (2018). Inter-annual rainfall variability may foster lake regime shifts: An example from Lake Bourget in France. *Ecological Modelling*, 389:11–18.

[Carpenter et al., 1999] Carpenter, S. R., Ludwig, D., and Brock, W. A. (1999). Management of Eutrophication for Lakes Subject to Potentially Irreversible Change. *Ecological Applications*, 9(3):751–771.

[de Groot, 2006] de Groot, R. (2006). Function-analysis and valuation as a tool to assess land use conflicts in planning for sustainable, multi-functional landscapes. *Landscape and Urban Planning*, 75(3):175 – 186.

[De Lara and Martinet, 2009] De Lara, M. and Martinet, V. (2009). Multi-criteria dynamic decision under uncertainty: A stochastic viability analysis and an application to sustainable fishery management. *Mathematical Biosciences*, 217(2):118–124.

[Désilles, 2020] Désilles, A. (2020). Viablab library. Technical report, LASTRES. https://github.com/lastre-viab/VIABLAB.

[Dowling et al., 2020] Dowling, N. A., Dichmont, C. M., Leigh, G. M., Pascoe, S., Pears, R. J., Roberts, T., Breen, S., Cannard, T., Mamula, A., and Mangel, M. (2020). Optimising harvest strategies over multiple objectives and stakeholder preferences. *Ecological Modelling*, 435:109243.

[Doyen, 2000] Doyen, L. (2000). Lipschitz Kernel of a Closed Set-Valued Map. *Set-Valued Analysis*, 8(1):101–109.

[Durand et al., 2017] Durand, M.-H., Désilles, A., Saint-Pierre, P., Angeon, V., and Ozier-Lafontaine, H. (2017). Agroecological transition: A viability model to assess soil restoration. *Natural resource modeling*, 30(3):e12134.

[Etienne, 2014] Etienne, M., editor (2014). *Companion Modelling: A Participatory Approach to Support Sustainable Development*. Springer Netherlands.

[Gajardo et al., 2017] Gajardo, P., Harmand, J., Ramirez, H., Rapaport, A., Riquelme, V., and Rousseau, A. (2017). Modeling and control of in-situ decontamination of large water resources. *ESAIM: Proceedings and Surveys*, 57:70–85.

[Hardin, 1968] Hardin, G. (1968). The Tragedy of the Commons. *Science, New Series*, 162(3859):1243–1248.

[Jacquet, 2018] Jacquet, S. (2018). Suivi scientifique du lac du Bourget pour l'année 2017. Technical report, INRAE ; CISALB ; Agence de l'eau RMC.

[Křivan, 1991] Křivan, V. (1991). Differential inclusions as a methodology tool in population biology. In *Proceedings of the 1995 European Simulation Multiconference*, pages 544–547, Prague.

[Křivan and Colombo, 1998] Křivan, V. and Colombo, G. (1998). A Non-stochastic Approach for Modeling Uncertainty in Population Dynamics. *Bulletin of Mathematical Biology*, 60(4):721–751.

[Lavallée, 2020] Lavallée, F. (2020). PhD thesis. Model, exploration, viability theory for management support of invasive species. Sorbonne University. In French.

[Little et al., 2007] Little, L. R., Punt, A. E., Mapstone, B. D., Pantus, F., Smith, A. D. M., Davies, C. R., and McDonald, A. D. (2007). ELFSim—A model for evaluating management options for spatially structured reef fish populations. *Ecological Modelling*, 205(3):381–396.

[Liu et al., 2015] Liu, W., Wang, S., Zhang, L., and Ni, Z. (2015). Water pollution characteristics of Dianchi Lake and the course of protection and pollution management. *Environmental Earth Sciences*, 74(5):3767–3780.

[Martin, 2004] Martin, S. (2004). The cost of restoration as a way of defining resilience: a viability approach applied to a model of lake eutrophication. *Ecol. Soc.*, 2(9):8.

[Martin and Alvarez, 2019] Martin, S. and Alvarez, I. (2019). Anticipating shocks in the state space: Characterizing robustness and building increasingly robust evolutions. *SIAM Journal on Control and Optimization*, 57(1):490–509.

[Martinet et al., 2016] Martinet, V., Peña-Torres, J., De Lara, M., and Ramírez C., H. (2016). Risk and Sustainability: Assessing Fishery Management Strategies. *Environmental and Resource Economics*, 64(4):683–707.

[Oubraham and Zaccour, 2018] Oubraham, A. and Zaccour, G. (2018). A Survey of Applications of Viability Theory to the Sustainable Exploitation of Renewable Resources. *Ecological Economics*, 145:346–367.

[Parrachino et al., 2006] Parrachino, I., Dinar, A., and Patrone, F. (2006). *Cooperative Game Theory And Its Application To Natural, Environmental, And Water Resource Issues : 3. Application To Water Resources.* Policy Research Working Papers. The World Bank.

[Saint-Pierre, 1994] Saint-Pierre, P. (1994). Approximation of the viability kernel. *Applied Mathematics and Optimization*, 29(2):187–209.

[Sanogo et al., 2012] Sanogo, C., Ben Miled, S., and Raissi, N. (2012). Viability Analysis of Multi-fishery. *Acta Biotheoretica*, 60(1):189–207.

[Team, 2010] Team, R. D. C. (2010). *R: A language and environment for statistical computing.* R Foundation for Statistical Computing, Vienna.

[Vinçon-Leite and Tassin, 1990] Vinçon-Leite, B. and Tassin, B. (1990). Modélisation de la qualité des eaux des lacs profonds : modèle thermique et biogéochimique du lac du Bourget. *La Houille Blanche*, (3-4):231–236.

[Vollenweider, 1982] Vollenweider (1982). coord. Eutrophication of waters. Monitoring, assessment and control. Technical report, OECD.

[Wei et al., 2013] Wei, W., Alvarez, I., and Martin, S. (2013). Sustainability analysis: Viability concepts to consider transient and asymptotical dynamics in socio-ecological tourism-based systems. *Ecological Modelling*, 251:103–113.

[Zaleski, 2020] Zaleski, L. (2020). PhD thesis. Decision and negociation assistant by viability analysis: Application to the participatory management of protected areas. Sorbonne University. In French.

A Properties of Viability Kernels

A.1 Properties of Multi-valued Maps

Let G be a multi-valued map from $\mathbb{R}^n$ to $\mathbb{R}^p$. The domain of G is $Dom(G) = \{x \in \mathbb{R}^n, G(x) \neq \emptyset\}$. The graph of G is $Graph(G) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p, y \in G(x)\}$. G has a linear growth if there is $c > 0$ such that for all $x \in Dom(G)$, $\|G(x)\| \leq c(\|x\| + 1)$. The system $Sc(f, U)$ (2) is Marchaud if f is continuous, $Graph(U)$ is closed, f and U have linear growth and the image set $\{f(x, u), u \in U(x)\}$ is convex for all of $x \in Dom(U)$. G is Lipschitz for constant $\lambda > 0$ (or λ-Lipschitz) if for all x_1, x_2 in $\mathbb{R}^n$, $G(x_1) \subset G(x_2) + \lambda\|x_1 - x_2\|B$, where B is the unit ball.

A.2 Closed Viability Kernels [Aubin, 1991]

Proposition 3 *Continuous case: When the system $Sc(f, U)$ (2) is Marchaud and K is closed, the associated viability kernel is closed. It is the largest viable set in K.*

Proposition 4 *Discrete case: When system $Sd(f, U)$ (3) is such that f is continuous, U has a linear growth, $Graph(U)$ is closed, and when K is closed, the associated viability kernel is closed.*

A.3 Guaranteed Viability Kernels

We recall some properties of dynamical controlled tychastic systems.

Definition 15 *A dynamical controlled tychastic system $Scv(f, U, V)$ (4) is Lipschitz if f is Lipschitz, and U and V are Lipschitz with compact images.*

Proposition 5 *From [Doyen, 2000] in the continuous case and [Lavallée, 2020] in the discrete case. The guaranteed viability kernel associated to a set K is closed when the dynamics verify the following conditions: In the continuous case, when K is closed and $Scv(f, U, V)$ is Lipschitz ; In the discrete case, when f and V are continuous, $Graph(U)$ is closed and U has a linear growth (see 2.2 for the definitions).*

We recall Theorem 1: The guaranteed viability kernel associated to K for $Scv(f, U, V)$ (with λ Lipschitz constant in the continuous case) (resp. $Sdv(f, U, V)$ in the discrete case) is a consensus solution to the management of $\mathcal{A}$.

Proof of Theorem 1. Let $L \neq \emptyset$ be the guaranteed viability kernel for system (4) with λ Lipschitz constant (resp. system (5) in the discrete case) associated to constraint set K. Let $\tilde{U}$ be the guaranteed regulation map. By definition, $Dom\tilde{U} = L$. We now prove that the guaranteed viability kernel is a viable set for each member i. Let $i \in \mathcal{N}$, we consider the system $Sc'_i = Sc(f_i, \tilde{U})$ (resp. $Sd'_i = Sd(f_i, \tilde{U})$ in the discrete case). Since the guaranteed viability kernel is defined from the group project (K, U), we have $L \subset K$, and for all $x \in L$, $\tilde{U}(x) \subset U(x)$. So an evolution governed by Sc'_i (resp. Sd'_i) is also an evolution governed by $Sc(f_i, U)$ (resp. $Sd(f_i, U)$). Since system (4) (resp. system (5)) embeds $Sc(f_i, U_i)$ (resp. $Sd(f_i, U_i)$), it also embeds Sc'_i (resp. Sd'_i). Let $x_0 \in L$, and let $x(.)$ (resp. x^k) be a trajectory starting at x_0 and governed by Sc'_i (resp. Sd'_i). Because of the embedding there is a function v such that $f_i(x(t), \tilde{u}(t)) = f(x(t), \tilde{u}(t), v(t))$ (resp. $f_i(x^k), \tilde{u}^k) = f(x^k, \tilde{u}^k, v^k)$ in the discrete case). So $x(.)$ is also an evolution governed by system (4) (resp.(5)). Since L is the guaranteed viability kernel for system (4) with λ Lipschitz constant (resp.(5)) associated to constraint K, from Definition (13), all trajectories starting from $x_0 \in L$ and governed by (4) (resp.(5)) are viable in L for control selection in $\tilde{U}$. So the trajectory of $x(.)$ governed by Sc'_i (resp. Sd'_i) starting at x_0 stays in L. So $x(.)$ is an evolution starting at x_0 governed by Sc_i (resp. Sd_i) viable in L. So L is a viable set for member i. ■

B Embedding system

Proposition 6 *If U has a linear growth and for all $i \in \mathcal{N}$, f_i has a linear growth, then a System (4) (resp. 5) can embed $Sc(f_i, U)$(2) (resp. $Sd(f_i, U)$ (3)) for all $i \in \mathcal{N}$.*

Proof. We note $M_x = \max_{i \in \mathcal{N}, u \in U(x)}(\|f_1(x,u) - f_i(x,u)\|)$. M_x is defined since U and all f_i have a linear growth. We define $f(x,u,v) = f_1(x,u) + v(x)$ with $v(x) \in V(x) = B(0, M_x)$, where $B(a,r)$ is the closed ball with center a and radius r. Then for $x \in K$ and $u \in U(x)$ we define $v_{i,u,x} = f_i(x,u) - f_1(x,u)$. We have $\|v_{i,u,x}\| \leq M_x$, so $v_{i,u,x} \in V(x)$. Then $\forall i \in \mathcal{N}, f_i(x,u) = f(x,u,v_{i,u,x})$ and equation (6) is verified. ■

Definition of f leading to smaller sets of perturbation are preferable. For instance, it can be interesting to define f with the convex hull of the $f_i(x,u)$: With $i \in J = \mathcal{N} \setminus \{1\}$ we consider $v = (v_i), v_i \in [0,1]$ with $\sum_{i \in J} v_i \leq 1$, and define $f(x,u,v) = f_1(x,u)(1 - \sum\limits_{i \in J} v_i) + \sum\limits_{i \in J} v_i f_i(x,u)$.